

## 5D black holes and non-linear sigma models

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**ABSTRACT:** Stationary solutions of 5D supergravity with  $U(1)$  isometry can be efficiently studied by dimensional reduction to three dimensions, where they reduce to solutions to a locally supersymmetric non-linear sigma model. We generalize this procedure to 5D gauged supergravity, and identify the corresponding gauging in 3D. We pay particular attention to the case where the Killing spinor is non constant along the fibration, which results, even for ungauged supergravity in 5D, in an additional gauging in 3D, without introducing any extra potential. We further study  $SU(2) \times U(1)$  symmetric solutions, which correspond to geodesic motion on the sigma model (with potential in the gauged case). We identify and study the algebra of BPS constraints relevant for the Breckenridge-Myers-Peet-Vafa black hole, the Gutowski-Reall black hole and several other BPS solutions, and obtain the corresponding radial wave functions in the semi-classical approximation.

**KEYWORDS:** Black Holes, Supergravity Models.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. 5D Black holes and non-linear sigma models</b>	<b>3</b>
2.1 Stationary solutions	4
2.2 Reduction to $\mathbb{R} \times U(1)$ symmetric solutions	5
2.3 Supersymmetry in 3 dimensions	7
2.4 $\mathbb{R} \times SU(2) \times U(1)$ symmetric solutions	12
<b>3. Supersymmetric solutions in <math>D = 5</math>, <math>\mathcal{N} = 1</math> minimal supergravity</b>	<b>14</b>
3.1 Geometry of $G_{2(2)}/SO(4)$	14
3.2 The BMPV and Taub-NUT black holes	20
3.3 The Gödel and Eguchi-Hanson black holes	23
3.4 The Gutowski-Reall black hole	26
3.5 Other solutions of gauged supergravity	29
<b>4. Discussion</b>	<b>30</b>
<b>A. More general nilpotent solutions of degree 2</b>	<b>31</b>

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## 1. Introduction

While mostly of theoretical interest, BPS solutions of 5D gravity and supergravity have been the subject of intense studies recently, partly due to their unexpected variety (see e.g. [1] for a recent review), to the relative simplicity of their microscopic dynamics [2, 3], to their relation to 4D BPS black holes via the 4D/5D lift [4, 5], and, in the case of gauged supergravity, to their relevance for the dynamics of four-dimensional gauge theories via the AdS/CFT correspondence. The purpose of this work is to develop algebraic techniques for constructing 5D BPS black hole solutions, generalizing existing methods which have been successfully applied to 4D black holes.

An important motivation for our study is the supersymmetric  $AdS_5$  black hole solution found by Gutowski and Reall (GR) [6, 7], which has remained particularly mysterious: for example, the solution exhibits a certain relation between the angular momentum and the electric charges, which is not implied by the  $N = 4$  superconformal algebra on the boundary. This restriction remains in generalizations involving two different angular momenta [8, 9]. This situation is in contrast with the asymptotically flat 5D space-time, where BPS solutions exist for arbitrary values of the angular momenta and charges within certain

bounds. It is an open problem whether more general  $AdS_5$  solutions exist where the above restriction is relaxed (see [10, 11] for recent progress on this issue).

Moreover, no microscopic counting of the entropy of the Gutowski-Reall black hole from the dual  $N = 4$  SYM theory is available to date. While 1/8-BPS (or more) supersymmetric states can be counted on the gauge theory side at weak coupling using a suitable index [12], 1/16-BPS states in general pair up due to the interactions, and the resulting (order  $N$ ) index is much smaller than the (order  $N^2$ ) entropy of a large GR black hole. Understanding this problem in more detail would be a useful step in bridging the gap between our remarkable control over black holes with high SUSY, and our qualitative understanding, at best, of non-supersymmetric black holes.

A heuristic model identifying a class of fermionic operators in the gauge theory which reproduces some of the scaling properties of the GR ( $AdS_5$ ) black hole was put forward in [13]. For 1/16-BPS black holes in  $AdS_4$  [14, 15], the same model suggests that the entropy counts  $N^{3/2}$  degrees of freedom. This may provide a supersymmetric setting to study the long-standing problem of counting the entropy of 2+1 dimensional strongly coupled fixed points.

In the flat four-dimensional case, the integrable structure of stationary solutions exposed by the dimensional reduction to three dimensions proved very useful in mapping the phase diagram of black holes. In this paper we generalize this algebraic description to stationary solutions of five-dimensional  $\mathcal{N} = 1$  supergravity with an additional  $U(1)$  symmetry. Thus, we assume two commuting Killing vectors  $\partial_t$  and  $\partial_\psi$ , time-like and space-like, respectively. The description in three dimensions is given in terms of a non-linear sigma model on (an analytic continuation of) a quaternionic-Kähler manifold  $\mathcal{M}_3$ , coupled to Euclidean gravity in three dimensions.

For explicitness, we focus for the most part on minimal supergravity in 5D, which leads to a non-linear sigma model on  $G_3/K_3 = G_{2(2)}/SO(4)$  (see [16, 17] for early discussions of this model, and [18, 19] for an independent study of its application to 5D black holes in ungauged supergravity). The same sigma model (up to analytic continuation) has appeared in the study of 4D black holes [20], and is in fact related to the present one by a flip of the  $t$  and  $\psi$  directions, corresponding to a Weyl reflection in a  $Sl(2)$  subgroup of  $G_3$ . The quaternionic geometry of  $G_{2(2)}/SO(4)$  was studied in great detail in [21], whose notations we follow.

A crucial difference with the dimensional reduction of 4D BPS black holes [23–26] is the fact that covariantly constant spinors in 5D are in general not constant along the orbit of the space-like Killing vector  $\partial_\psi$ . As a result, the 3D sigma model including the fermions is gauged, even when the 5D supergravity is ungauged. This gauging does not affect the bosonic part of the action, however. When the 5D supergravity is gauged, an additional potential is generated in the 3D sigma model. We identify the correct gauging in the general framework for 3D gauged supergravities laid out in [27, 28].

When a further  $SU(2)$  symmetry is present, the model may be further reduced to one dimension, where it reduces to geodesic motion of a fictitious particle on a real cone over  $\mathcal{M}_3$ . The gauging in 5D introduces a potential, which spoils the integrability of the model in general. Some of the algebraic structure however does carry over, and determines the structure of the BPS constraints.

As a by-product of this analysis, we obtain the semi-classical form of the radial wave function for 5D BPS black holes, i.e. the solution to the BPS constraints in Hamilton-Jacobi formalism.

The outline of the paper is as follows. In section 2 we discuss the dimensional reduction of 5D  $\mathcal{N} = 1$  supergravity down to 4, 3 and 1 dimensions. We identify the gauging of the 3D sigma model coming from the dependence of the Killing spinor along the direction  $\psi$ , and from the gauging in 5 dimensions. In section 3 we specialize to stationary solutions with  $U(1) \times SU(2)$  isometries in minimal supergravity in 5D. We identify the algebraic structure of the supersymmetry constraints appropriate to the BMPV, Taub-NUT, Gödel, Eguchi-Hanson and Gutowski-Reall black holes, respectively, and compute their respective Noether charges and radial wave functions. Section 4 contains a brief summary and discussion. A particular class of solutions with nilpotent Noether charges of degree 2 is presented in appendix A.

## 2. 5D Black holes and non-linear sigma models

We consider  $\mathcal{N} = 1$ ,  $D = 4 + 1$  supergravity coupled to  $n$  abelian vector multiplets. The bosonic part of the Lagrangian is given by

$$e^{-1} \mathcal{L}_5 = -\frac{1}{2}R - \frac{1}{4}G_{IJ}F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2}g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{e^{-1}}{48}\epsilon^{\mu\nu\rho\sigma\lambda}C_{IJK}F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K \quad (2.1)$$

The scalars  $\varphi^i$  take value in the moduli space  $\mathcal{M}_5$ , given by the cubic hypersurface

$$I_3(h) \equiv \frac{1}{6}C_{IJK}h^I h^J h^K = 1 \quad (2.2)$$

where  $C_{IJK}$  are constants. The metric for the kinetic terms of the scalars  $\varphi^i$  and the gauge fields  $A^I$  are given by

$$g_{ij} = G_{IJ}\partial_{\varphi^i}h^I\partial_{\varphi^j}h^J, \quad G_{IJ} = -\frac{1}{2}\partial_{h^I}\partial_{h^J}\log I_3(h), \quad (2.3)$$

evaluated on the hypersurface (2.2). For simplicity, we restrict to the case where the moduli space is a symmetric space, so that [29]

$$C_{IJK}C_{J'(LM}C_{PQ)K'}\delta^{JJ'}\delta^{KK'} = \frac{4}{3}\delta_{I(L}C_{MNP)} \quad (2.4)$$

In this case  $I_3(h)$  is the norm form of a Jordan algebra  $J$  of degree 3 [30],  $\mathcal{M}_5 = G_5/K_5 = \text{Str}_0(J)/\text{Aut}(J)$  where  $\text{Str}_0(J)$  and  $\text{Aut}(J)$  are the reduced structure group and automorphism groups of  $J$ , and

$$C^{IJK} \equiv \delta^{II'}\delta^{JJ'}\delta^{KK'}C_{I'J'K'}. \quad (2.5)$$

It is useful to define the ‘‘adjoint map’’

$$h_I^\sharp \equiv \frac{1}{6}C_{IJK}h^J h^K, \quad (2.6)$$

which satisfies

$$h^I = \frac{9}{2}C^{IJK}h_J^\sharp h_K^\sharp \quad (2.7)$$

In cases where the moduli space is not symmetric, the reduction procedure that we shall describe below still applies, however the resulting moduli space in three dimensions is no longer symmetric.

In the absence of hypermultiplets, it is possible to include a Fayet-Iliopoulos term for a linear combination  $\mathcal{A}_\mu = V_I A_\mu^I$  of the  $n$  gauge fields ( $V_I$  are numerical constants). The Lagrangian becomes  $e^{-1}\mathcal{L}_{5;\text{gauged}} = e^{-1}\mathcal{L}_5 + V_5$ , where the potential is given by [30, 31]

$$V_5 = g^2 V_I V_J \left( 6h^I h^J - \frac{9}{2} g^{ij} \partial_i h^I \partial_j h^J \right) = 27 C^{IJK} V_I V_J h_K \quad (2.8)$$

The potential admits an  $AdS_5$  vacuum provided  $V_I$  lies inside the cone  $I_3(V) > 0$ .

### 2.1 Stationary solutions

Assuming the existence of a time-like Killing vector, the five-dimensional metric and gauge fields can be taken in the form

$$ds_5^2 = -f^2(dt + \omega_4)^2 + f^{-1}ds_4^2, \quad A_5^I = \phi^I(dt + \omega_4) + A_4^I \quad (2.9)$$

where  $f$ ,  $\phi^I$  and  $\varphi^i$  are independent of time, and  $A_4^I, \omega_4$  are one-forms on the four-dimensional Euclidean slice. The equations of motion for this ansatz are most easily obtained by reducing the Lagrangian along the time direction. This leads to  $\mathcal{N} = 2$  supergravity in  $D = 4$  Euclidean dimensions, coupled to  $n + 1$  vector multiplets. The reduced Lagrangian  $\mathcal{L}_4$  is determined in the usual way by the holomorphic prepotential

$$F = \frac{1}{6} C_{IJK} X^I X^J X^K / X^0 \quad (2.10)$$

In contrast to the usual Kaluza-Klein reduction along a space-like direction, studied for example in [32], the special coordinates  $z^I = X^I / X^0$  and  $\bar{z}^I = \bar{X}^I / \bar{X}^0$  are independent real variables,

$$z^I = \frac{X^I}{X^0} = \phi^I + \mathcal{I} f h^I, \quad \bar{z}^I = \phi^I - \mathcal{I} f h^I \quad (2.11)$$

where  $\mathcal{I}^2 = -1, \bar{\mathcal{I}} = -\mathcal{I}$  is a “para-complex” structure [33]. As a result, the vector moduli space  $\mathcal{M}_4^*$  has split signature  $(n + 1, n + 1)$ . In the following, we will perform an analytic continuation  $\phi^I \rightarrow i\phi^I$ , which allows us to work with the standard complex structure  $\mathcal{I} = i$ , albeit with a purely imaginary  $\phi^I$ . Similarly, we shall continue  $\omega_4 \rightarrow i\omega_4$ , so that  $A_4^\Lambda = (A_4^0, A_4^I) = (\omega_4, A_4^I)$  and their magnetic duals transform as a vector of  $Sp(2n + 2)$ . For later reference, we note that the Kähler potential is given by

$$K = -\log I_3(z^I - \bar{z}^I) = -3 \log f \quad (2.12)$$

When  $I_3$  is the norm form of a Jordan algebra  $J$ , the vector multiplet moduli space is a symmetric space  $\mathcal{M}_4 = G_4 / K_4 = \text{Conf}(J) / [U(1) \times \widehat{\text{Str}}_0(J)]$ , where  $\text{Conf}(J)$  is the conformal group of  $J$  and  $\widehat{\text{Str}}_0(J)$  is the compact form of the reduced structure group of  $J$ .

In the presence of Fayet-Iliopoulos terms in 5 dimensions, the scalar potential (2.8) leads to a potential  $V_4$  in four dimensions,

$$V_4 = f^{-1} V_5 \quad (2.13)$$

Using for example the identities found in [34], one may check that (2.13) is consistent with the general form of the scalar potential induced by Fayet-Iliopoulos terms in four dimensions [35, 36],

$$V_4 = g^2 \left( g^{i\bar{j}} f_i^\Lambda f_{\bar{j}}^\Sigma - 3e^K \bar{X}^\Lambda X^\Sigma \right) \vec{P}_\Lambda \cdot \vec{P}_\Sigma \quad (2.14)$$

where  $\vec{P}_\Lambda$  are the triplets of Fayet-Iliopoulos terms, chosen as

$$\vec{P}_0 = 0, \quad \vec{P}_I = V_I \vec{n}, \quad \vec{n} \cdot \vec{n} = 1. \quad (2.15)$$

In the language of  $\mathcal{N} = 1$  supergravity, this corresponds to a superpotential  $W = g V_I X^I$ .

## 2.2 Reduction to $\mathbb{R} \times \text{U}(1)$ symmetric solutions

We now restrict to solutions with a extra  $\text{U}(1)$  isometry, generated by a Killing vector  $\partial_\psi$  on the four-dimensional spatial slice. Accordingly, the spatial metric  $ds_4^2$  decomposes as

$$ds_4^2 = e^{2U} (d\psi + \omega_3)^2 + e^{-2U} ds_3^2 \quad (2.16)$$

while the gauge fields decompose as

$$A_4^\Lambda = \zeta^\Lambda (d\psi + \omega_3) + A_3^\Lambda \quad (2.17)$$

The equations of motion for this ansatz can be obtained by further reducing the four-dimensional Euclidean supergravity along the space-like direction  $\partial_\psi$ . Upon dualizing the gauge fields  $A_3^\Lambda$  and  $\omega_3$  into pseudo-scalars  $\tilde{\zeta}_\Lambda$  and  $\sigma$ , one obtains  $\mathcal{N} = 4$  supergravity in three Euclidean dimensions coupled to a non-linear sigma model, with Lagrangian [38]<sup>1</sup>

$$e^{-1} \mathcal{L}_3 = -\frac{1}{2} R - \frac{1}{2} G_{ab} \partial\varphi^a \partial\varphi^b + V_3 \quad (2.18)$$

The scalar potential  $V_3$ , present only in the case of gauged supergravity, is given by the reduction of (2.13),

$$V_3 = f^{-1} e^{-2U} V_5 \quad (2.19)$$

The target space of the sigma model, which we shall denote by  $\mathcal{M}_3$ , is coordinatized by<sup>2</sup>  $\varphi^a = \{U, z^I, \bar{z}^{\bar{I}}, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma\}$ . It is related to the more familiar quaternionic-Kähler manifold  $\mathcal{M}_{3,E}$  (known as the  $c$ -map of  $\mathcal{M}_4$ ) arising in the usual Kaluza-Klein reduction along space-like directions, and with positive-definite metric [42]

$$ds_{\mathcal{M}_{3,E}}^2 = dU^2 + g_{I\bar{J}} dz^I d\bar{z}^{\bar{J}} + e^{-4U} \left( d\sigma - \tilde{\zeta}_\Lambda d\zeta^\Lambda + \zeta^\Lambda d\tilde{\zeta}_\Lambda \right)^2 - \frac{1}{2} e^{-2U} \left[ (\text{Im}\mathcal{N})_{\Lambda\Sigma} d\zeta^\Lambda d\zeta^\Sigma + (\text{Im}\mathcal{N})^{\Lambda\Sigma} \left( d\tilde{\zeta}_\Lambda + (\text{Re}\mathcal{N})_{\Lambda R} d\zeta^R \right) \left( d\tilde{\zeta}_\Sigma + (\text{Re}\mathcal{N})_{\Sigma T} d\zeta^T \right) \right], \quad (2.20)$$

by analytically continuing

$$\left( \phi^I, \zeta^0, \zeta^I, \tilde{\zeta}_I, \tilde{\zeta}_0, \sigma \right) \rightarrow \left( i\phi^I, i\zeta^0, \zeta^I, i\tilde{\zeta}_I, \tilde{\zeta}_0, i\sigma \right). \quad (2.21)$$

<sup>1</sup>The full supersymmetry in the gauged case can only be displayed by adding two auxiliary gauge fields with Chern-Simons couplings, see section 2.3 below.

<sup>2</sup>The symbol  $\varphi$ , used in (2.1) to denote the scalars in 5 dimensions, hereforth denotes the scalars in 3 dimensions.

For convenience, we shall be using the Riemannian metric (2.20), but allow  $\phi^I, \zeta^0, \tilde{\zeta}_I, \sigma$  to be purely imaginary. We note that an equally valid procedure would have been to perform the reduction along the space-like Killing vector  $\partial_\psi$  first, and then along the time-like Killing vector  $\partial_t$ . This of course leads to the same non-linear sigma model on  $\mathcal{M}_3$  in three Euclidean dimensions, however the analytic continuation that relates it to the Riemannian manifold  $\mathcal{M}_{3,E}$ , in the variables appropriate to this reduction, is now

$$\left(\phi^I, \zeta^0, \zeta^I, \tilde{\zeta}_I, \tilde{\zeta}_0, \sigma\right) \rightarrow \left(\phi^I, i\zeta^0, i\zeta^I, i\tilde{\zeta}_I, i\tilde{\zeta}_0, \sigma\right). \quad (2.22)$$

As we discuss later in this section, the two analytic continuations (2.21) and (2.22) are in fact related by a Weyl reflection in an  $Sl(2)$  subgroup of their isometry group, corresponding to the exchange of the  $t$  and  $\psi$  direction. It is also worthwhile to note that the same sigma-model arises when describing stationary solutions in  $D = 3+1$   $\mathcal{N} = 2$  supergravity [37, 23–26]; as we shall see however, the supersymmetry conditions corresponding to 5D black holes differ from those pertaining to 4D black holes.

In general, the space  $\mathcal{M}_3$  admits a solvable algebra of isometries, originating from the diffeomorphism and gauge symmetries in 5 dimensions: the Killing vectors

$$p^\Lambda = \partial_{\tilde{\zeta}_\Lambda} + \zeta^\Lambda \partial_\sigma, \quad q_\Lambda = \partial_{\zeta^\Lambda} - \tilde{\zeta}_\Lambda \partial_\sigma, \quad k = \partial_\sigma \quad (2.23a)$$

generate a Heisenberg algebra  $[p^\Lambda, q_\Sigma] = -2\delta_\Sigma^\Lambda k$ ; the generators

$$T_I = \partial_{\phi^I} + \zeta^0 \partial_{\zeta^I} - C_{IJK} \zeta^J \partial_{\tilde{\zeta}_K} - \tilde{\zeta}_I \partial_{\tilde{\zeta}_0} \quad (2.23b)$$

are nilpotent of degree 4, act symplectically on  $(p^\Lambda, q_\Lambda)$ , and commute with  $k$ ; the non-compact generators

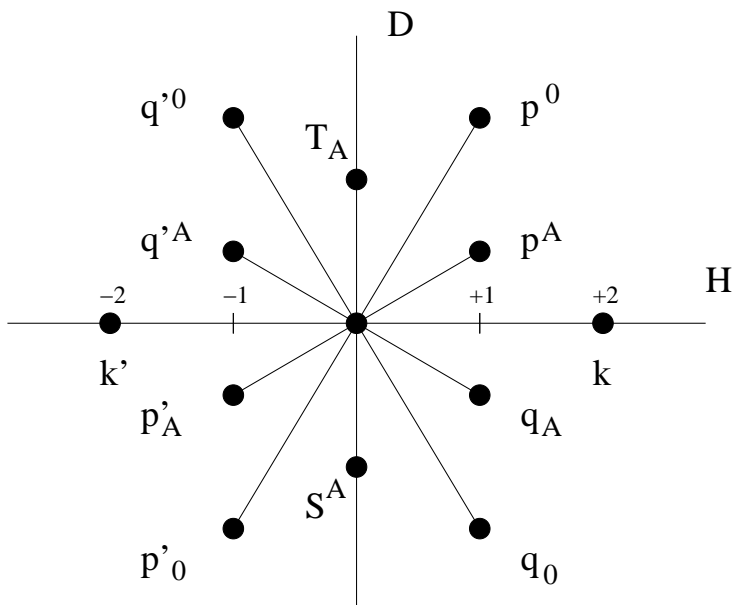
$$H = -\partial_U - \zeta^\Lambda \partial_{\zeta^\Lambda} - \tilde{\zeta}_\Lambda \partial_{\tilde{\zeta}_\Lambda} - 2\sigma \partial_\sigma, \quad (2.23c)$$

$$D = -\frac{1}{2} \left( -3\zeta^0 \partial_{\zeta^0} - \zeta^I \partial_{\zeta^I} + \tilde{\zeta}_I \partial_{\tilde{\zeta}_I} + 2\phi^I \partial_{\phi^I} + 2t^I \partial_{t^I} + 3\tilde{\zeta}_0 \partial_{\tilde{\zeta}_0} \right) \quad (2.23d)$$

give a bi-grading of the nilpotent part of the algebra. The presence of the potential  $V_3$  breaks the  $(H, D)$  symmetry to  $H - 2D$ , but leaves all other generators above unbroken.

When  $I_3$  is the norm form of a Jordan algebra  $J$ , the solvable group of isometries is extended to a semi-simple group  $\text{QConf}(J)$ , such that  $\mathcal{M}_3 = G_3/K_3 = \text{QConf}(J)/\text{SU}(2)_L \times \widetilde{\text{Conf}}(J)$  becomes a quaternionic-Kähler symmetric space. Here  $\text{QConf}(J)$  is the “quasi-conformal group” associated to  $J$  [22, 21] (see e.g. [26] for a review). It is obtained by supplementing the above generators with special transformations  $S^J$  and rotations  $R^J$ , such that  $\{T_I, S^J, R^J, D\}$  generate  $G_4 = \text{Conf}(J)$ , and with a “dual” Heisenberg algebra,  $[p'_\Lambda, q'^\Sigma] = -2\delta_\Lambda^\Sigma k'$ , requiring that  $\{k', H, k\}$  generate  $Sl(2, \mathbb{R})$ . The  $\text{SU}(2)_L$  factor in the maximal compact subgroup  $K_3$  is the first factor in the R-symmetry group  $\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$ , the scalars being inert under the second factor  $\text{SU}(2)_R$ , which would act on the hypermultiplets if those were present [38].

For later purposes, it will be useful to recall that the root diagram of  $G_3 = \text{QConf}(J)$  admits a two-dimensional projection given by the root diagram of the exceptional group  $G_2$ , where the long roots have multiplicity one and the short roots have multiplicity  $n + 1$



**Figure 1:** Two-dimensional projection of the root diagram of  $G_3 = \text{QConf}(J)$  with respect to the split Cartan torus  $(H, D)$ . The long roots have multiplicity 1, while the short roots have multiplicity  $n + 1$ .

(see figure 1). In particular, for minimal supergravity in 5 dimensions, with  $I_3(h) = h^3$  and  $n = 0$ , the group  $G_3$  is in fact  $G_{2(2)}$  itself. The long roots in figure 1 generate a  $Sl(3, \mathbb{R})$  subgroup of  $G_3$ , which is the symmetry arising in the dimensional reduction of pure Einstein gravity in 5 dimensions down to 3 dimensions [39–41]. In particular, the  $Sl(2, \mathbb{R})$  subgroup generated by the roots  $q_0, q'^0$  and their commutator is the symmetry exchanging the time-like and space-like Killing vectors  $\partial_t$  and  $\partial_\psi$ , alluded to below (2.22). The  $Sl(2, \mathbb{R})$  subgroup generated by  $k, k'$  and their commutator instead corresponds to the Ehlers symmetry of Einstein gravity in 4 dimensions. In the presence of the potential  $V_3$ , the only unbroken symmetries are  $p^\Lambda, q_\Lambda, k$  and  $T_I, q'_0$ . As we shall see shortly, the conserved charges associated to the latter are the electric charges and angular momentum of the 5D black hole.

### 2.3 Supersymmetry in 3 dimensions

In the absence of gauging, the supersymmetry of the  $N = 4$  sigma model coupled to gravity in three dimensions was discussed in [38]. When gravity is gauged in 5 dimensions or when the spinors are non-trivial along the fibers, then we obtain a gauged model in three dimensions. In this subsection we will review some aspects of gauged sigma models in three dimensions, which we will use in subsequent sections. We will mainly follow the discussion in [27] and [28].

**The ungauged case.** A locally supersymmetric  $\mathcal{N} = 4$  sigma model in 3 dimensions has an  $SO(4) \sim SU(2)_L \times SU(2)_R$  R-symmetry. Out of this symmetry group,  $SU(2)_R$  is already apparent in 5 dimensions where it acts on hypermultiplets, leaving the bosonic fields in the



vector multiplets inert. The other factor  $SU(2)_L$  is manifest only when reducing to 3D and is associated to rotations in the two-plane of the fiber. We use the following notations: a vector of the  $SO(4)$  R-symmetry carries an index  $I$ , which is equivalent to a bi-spinor  $\alpha\dot{\alpha}$  ( $\alpha = 1, 2, \dot{\alpha} = 1, 2$ ), where the dotted index is the index for  $SU(2)_L$ . Indices of the adjoint of  $SO(4) \equiv SU(2)_L \times SU(2)_R$  will occasionally be denoted by  $x = 1, 2, 3$  and  $\dot{x} = 1, 2, 3$ . In addition,  $a, b, \dots$  will denote indices of coordinates on the manifold. The  $SO(4)$  R-symmetry determines an  $SO(4)$  connection, denoted by  $Q_a^{[IJ]}$ , or  $Q_a^{\dot{x}}$  and  $Q_a^x$ . Our case is more special — since there are no hypermultiplets in 5 dimensions, one of the  $SU(2)$  does not act on the bosons and hence  $Q^x = 0$ .

The variations of the gravitini and hyperini (in a vanishing fermionic background) are then given by

$$\delta\psi_\mu^{\alpha\dot{\alpha}} = \left( D_\mu \epsilon_{\dot{\alpha}\beta} + Q_a^{\dot{x}} \sigma_{\dot{\alpha}\beta}^{\dot{x}} \partial_\mu \varphi^a \right) \eta_{\alpha\dot{\beta}} \tag{2.24}$$

$$\delta\chi^{A\alpha} = V_a^{A\dot{\alpha}} \eta_{\alpha\dot{\alpha}} \tag{2.25}$$

where  $\eta_{\alpha\dot{\alpha}}$  is the supersymmetry parameter, and  $V_a^{A\dot{\alpha}}$  is the quaternionic viel-bein, related to the metric  $G_{ab}$  and the quaternionic-Kähler forms  $\Omega_{ab}^{\dot{x}}$  by

$$G_{ab} = V_a^{A\dot{\alpha}} \epsilon_{\dot{\alpha}\beta} \Sigma_{AB} V_b^{B\dot{\beta}}, \quad \Omega_{ab}^{\dot{x}} = V_a^{A\dot{\alpha}} \sigma_{\dot{\alpha}\beta}^{\dot{x}} \Sigma_{AB} V_b^{B\dot{\beta}} \tag{2.26}$$

where  $\Sigma_{AB}$  is the  $Sp(2n+2)$  invariant antisymmetric tensor. The quaternionic viel-bein  $V^{A\dot{\alpha}}$  is a  $2 \times (2n+2)$  matrix, which was computed for the c-map metric (2.20) in [42]:

$$V^{A\dot{\alpha}} = \begin{pmatrix} \bar{u} & \bar{v} \\ -i\bar{e}_i & \bar{E}_i \\ E_i & -ie_i \\ -v & u \end{pmatrix}, \tag{2.27}$$

where

$$u = e^{-U+K/2} X^\Lambda \left( d\tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Sigma} d\zeta^\Sigma \right) \tag{2.28a}$$

$$v = dU - \frac{i}{2} e^{-2U} \left( d\sigma + \zeta^\Lambda d\tilde{\zeta}_\Lambda - \tilde{\zeta}_\Lambda d\zeta^\Lambda \right) \tag{2.28b}$$

$$E_i = i e^{-U} e_{i\bar{i}} g^{i\bar{j}} \bar{f}_{\bar{j}}^\Lambda \left( d\tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Sigma} d\zeta^\Sigma \right). \tag{2.28c}$$

The expression for  $e_i$  is too cumbersome and will be given only for the special case that we will be interested in below. In general, however,  $e_i = e_{Ii} dz^I$  is the holomorphic viel-bein of the special Kähler manifold  $\mathcal{M}_4$ .

**The gauged case — some general formulas.** In the presence of gauging in 5 dimensions, the  $\mathcal{N} = 4$  sigma model in three dimensions has to be gauged. Our main goal in the rest of this subsection is to identify the appropriate gauging that corresponds to the potential (2.19).<sup>3</sup>

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<sup>3</sup>We are grateful to H. Samtleben for invaluable advice about gaugings of three-dimensional supergravities.

There are different equivalent ways of writing gauged sigma models in three dimensions. While all the massless dynamical bosonic degrees of freedom have already been accounted for in the reduction, it is possible to add auxiliary gauge fields which have no Maxwell-type kinetic terms, but have Chern-Simons terms (in addition to their couplings to matter). In fact, in three dimensions a single gauge field with a standard Maxwell kinetic term may be replaced by two gauge fields and a scalar [43]. The two gauge fields are non-dynamical, and coupled by a Chern-Simons interaction. If there are no massless charged fields in the original Lagrangian, the two new gauge fields can be integrated out, and the remaining scalar is just the dual of the original Maxwell field. Otherwise, they need to be kept in the action since these two gauge fields still couple to matter fields via a  $A_\mu J^\mu$  coupling which cannot be dualized. Note that in this formalism, vector fields always appear in pairs.

There are two sources of gauging and Chern-Simons vector fields in 3D. One is the gauging in 5D, which generates a potential in 5D and hence a potential in 3D. The other source is the dependence of the fermionic fields on the fibers in the reduction from 5D to 3D. In order to make the physical degrees of freedom apparent, our aim is to dualize all vector fields into scalars. This includes off-diagonal components of the 5D metric, which become vector fields in 3D. The latter may be dualized into scalars provided there are no field charged under these gauge field. The bosonic fields are always invariant under translation in the fiber directions, hence are never charged. In contrast, the 3D fermions may or may not be constant in the directions of the fiber. If they are not constant then the fermions are charged under these vector fields and the sigma model will be inherently gauged. Note that by itself, this second source of gauging never generates a potential in 3D, since only the behavior of the fermion is affected. It is therefore akin to the “no scale supergravity models” familiar in higher-dimensional supergravity [44, 30].

Both sources of the gauging, provided they preserve  $\mathcal{N} = 4$  supersymmetry, may be described in the same formalism [27, 28]. The global symmetries acting on the fields in the sigma model are  $G_3 \times \text{SU}(2)_L \times \text{SU}(2)_R$ , where  $G_3$  is the isometry group of the manifold  $\mathcal{M}_3$ , and  $\text{SU}(2)_L \times \text{SU}(2)_R$  is the R-symmetry group. Gaugings are classified by a symmetric tensor  $\Theta_{\mathcal{MN}}$ , which encodes the embedding of the gauged symmetry group into the global symmetry group. Here  $\mathcal{M}$  and  $\mathcal{N}$  live in the Lie algebra of symmetries. The fact that  $\Theta$  is symmetric is related to the pairing of Chern-Simons vector fields discussed above. We shall denote by  $T^{\mathcal{M}} = \{T^m, S^{\dot{x}}, S^x\}$ ,  $m (n, p, \dots) = 1, \dots, \dim G_3$ ,  $x (y, z) = 1, 2, 3, \dot{x} (\dot{y}, \dot{z}) = 1, 2, 3$  a basis of this Lie algebra. In the case (of interest in this paper) where only the mixed components  $\Theta_{mx}$  are non-zero, the condition that  $\Theta_{\mathcal{MN}}$  be invariant under the gauge group (eq. 3.13 in [27]) reduces to

$$\epsilon^{xyz} \Theta_{mx} \Theta_{ny} = 0, \quad f_p^{mn} \Theta_{mx} \Theta_{ny} = 0. \tag{2.29}$$

These conditions are equivalent to the requirement that  $\Theta$  be decomposable,

$$\Theta_{mx} = V_m n_x, \tag{2.30}$$

where  $n_x$  is a vector in  $\mathbb{R}^3$  and  $V_m T^m$  an element in  $\mathfrak{g}_3$ . Thus, this choice of projection tensor corresponds to a rank 2 abelian gauge group  $U(1) \times U(1)$ , corresponding to the generators

$V_m T^m$  in  $G_3$  and  $n_x S^x$  in  $SU(2)_R$ , respectively. Accordingly, one should introduce two Abelian gauge fields  $A_\mu$  and  $B_\mu$  in three dimensions, with Chern-Simons coupling  $AdB$ .

The other ingredient is the moment map  $\mathcal{V}^{\mathcal{M};[IJ]} = \mathcal{X}^{\mathcal{M},a} Q_a^{[IJ]} + S^{\mathcal{M},[IJ]}$ . Here,  $\mathcal{X}^{\mathcal{M},a} \partial_{\varphi^a}$  is the vector field on the quaternionic-Kähler space  $\mathcal{M}_3$  corresponding to the symmetry generator  $T^{\mathcal{M}}$ ,  $Q_a^{[IJ]} d\varphi^a$  is the  $SO(4)$  connection on  $\mathcal{M}_3$ , and  $S^{\mathcal{M},[IJ]}$  is the compensating R-symmetry induced by the action of  $T^{\mathcal{M}}$ . Rewriting the antisymmetric pair of indices  $[IJ]$  as either  $x$  or  $\dot{x}$ , it is clear that

$$\mathcal{V}^{x;y} = \delta^{xy}, \quad \mathcal{V}^{x;\dot{x}} = 0, \quad \mathcal{V}^{m;x} = 0, \quad (2.31)$$

while  $\mathcal{V}^{m;\dot{x}}$  is the usual moment map of quaternionic isometries, defined by [45]

$$d\mathcal{V}^{m;\dot{x}} + \epsilon^{\dot{x}\dot{y}\dot{z}} Q^{\dot{y}} \wedge \mathcal{V}^{m;\dot{z}} = \mathcal{X}^m \cdot \Omega^{\dot{x}} \quad (2.32)$$

where  $\Omega^{\dot{x}}$  is the triplet of quaternionic two-forms and  $Q^{\dot{x}}$  is the  $SU(2)_L$  connection.

With the embedding tensor  $\Theta_{\mathcal{MN}}$  and the moment map  $\mathcal{V}^{\mathcal{M};[IJ]}$  at hand, one can construct the  $T$ -tensors

$$T^{IJ,KL} = \mathcal{V}^{\mathcal{M};[IJ]} \Theta_{\mathcal{MN}} \mathcal{V}^{\mathcal{N};[KL]}, \quad T^{IJ,a} = \mathcal{V}^{\mathcal{M};[IJ]} \Theta_{\mathcal{MN}} \mathcal{V}^{\mathcal{N};a} \quad (2.33)$$

where  $\mathcal{V}^{\mathcal{M};a} \equiv \mathcal{X}^{\mathcal{M},a}$ , and the  $A$ -tensors

$$A_1^{IJ} = \frac{1}{3} T^{MN,MN} \delta^{IJ} - 2T^{IL,JL}, \quad A_{2,a}^{IJ} = \frac{1}{4} (D_a A_1^{IJ} + 2T_a^{IJ}) \quad (2.34)$$

Using (2.31), we see that the only non-vanishing components of the  $T$ -tensors are

$$T^{x,\dot{x}} = \mathcal{V}^{\dot{x}} n^x, \quad T^{x,a} = \mathcal{X}^a n^x, \quad \mathcal{V}^{\dot{x}} \equiv V_m \mathcal{V}^{m;\dot{x}}, \quad \mathcal{X}^a \equiv V_m \mathcal{X}^{m,a} \quad (2.35)$$

Therefore, the non-vanishing components of the  $A$  tensors are

$$A_1^{x\dot{x}} = \mathcal{V}^{\dot{x}} n^x, \quad A_{2,a}^{x\dot{x}} = \frac{1}{4} \mathcal{X}^b \Omega_{ab}^{\dot{x}} n^x, \quad A_{2,a}^x = \mathcal{X}_a n^x \quad (2.36)$$

Finally, the scalar potential is obtained from

$$V_3 = -\frac{g^2}{4} \left( A_1^{IJ} A_1^{IJ} - 2g^{ab} A_{2,a}^{IJ} A_{2,b}^{IJ} \right) = -\frac{g^2}{4} \left( \mathcal{V}^2 - \frac{1}{4} \mathcal{X}^a g_{ab} \mathcal{X}^b \right) \quad (2.37)$$

Using the formulas for  $A_1$  and  $A_2$  for this specific gauging, and using eq. 6.7 of [28], we find that the supersymmetry variation of the gravitini and the hyperini are:

$$\delta\psi_\mu^{\alpha\dot{\alpha}} = \left[ \mathcal{V}^{\dot{x}} \sigma_{\dot{\alpha}\dot{\beta}}^{\dot{x}} (\epsilon_{\alpha\beta} B_\mu + \sigma_{\alpha\beta}^x u^x \gamma_\mu) + \epsilon_{\dot{\alpha}\dot{\beta}} \sigma_{\alpha\beta}^x u^x A_\mu \right] \eta_{\beta\dot{\beta}} \quad (2.38)$$

$$\delta\chi^{A\alpha} = V_a^{A\dot{\alpha}} [(\partial_\mu \phi^a + g \mathcal{X}^a B_\mu) \gamma^\mu + g \mathcal{X}^a] \eta_{\alpha\dot{\alpha}} \quad (2.39)$$

where  $\eta_{\beta\dot{\beta}}$  is the supersymmetry parameter, and  $V_a^{A\dot{\alpha}} d\varphi^a$  is the quaternionic vielbein (2.27).

**The gauging unmasked.** Our remaining task is to identify the Killing vector  $\mathcal{X}$  underlying the scalar potential (2.19). The result depends on the source of gauging:

- (i) *Gauging in 5 dimensions.* An important constraint is that the Killing vector  $\mathcal{X}$  must commute with the action of the  $Sl(2)$  symmetry exchanging the  $\partial_t$  and  $\partial_\psi$  Killing vectors. Moreover, it should commute with the Heisenberg generators. These constraints uniquely determine

$$\mathcal{X} = V_I \left( \partial_{\tilde{\zeta}_I} - \zeta^I \partial_\sigma \right), \tag{2.40}$$

where the  $V_I$  are the coefficients that determine the F-I term and the 5D potential in (2.8).

The  $SU(2)$  connection on (2.20) being given by [42]

$$Q^{\dot{3}} = e^{-2U} \left( d\sigma + \zeta^\Lambda d\tilde{\zeta}_\Lambda - \tilde{\zeta}_\Lambda d\zeta^\Lambda \right) + \frac{1}{4} Q_K, \quad Q^{\dot{1}} = \text{Re}(u), \quad Q^{\dot{2}} = \text{Im}(u) \tag{2.41}$$

where  $Q_K = (\partial_{z^i} K dz^i - \partial_{\bar{z}^{\bar{i}}} K d\bar{z}^{\bar{i}})/2i$  is the Kähler connection on  $\mathcal{M}_4$ , we can compute the moment maps

$$\mathcal{V}^{\dot{3}} = e^{-2U} V_\Lambda \zeta^\Lambda, \quad \mathcal{V}^{\dot{1}} = e^{-2U} V_\Lambda \text{Re}(X^\Lambda), \quad \mathcal{V}^{\dot{2}} = e^{-2U} V_\Lambda \text{Im}(X^\Lambda). \tag{2.42}$$

Plugging into (2.37), and using identities in [34], we find that the scalar potential (2.37) does indeed reproduce (2.19),

$$V_3 = f^{-1} e^{-2U} V_5. \tag{2.43}$$

- (ii) *Gauging due to compactification.* The standard dimensional reduction from 5D to 3D assumes that the fermions are constant along the fibers. Some of the solutions that we will discuss however, have a more complicated variation along the fibers [46]. In such a case, the fermions are charged under the Maxwell gauge field associated to the off-diagonal metric in the fiber directions, which prevents its dualization into a pseudo-scalar. Thus, one obtains a pair of  $U(1)$  gauge fields for every shift isometry under which the fermions are charged. Since the bosonic action is the same, irrespective of the charge of the fermions, it had better be the case that this gauging does not change the potential (and in particular, it does not introduce a potential if there was none to start with). Indeed, it is possible to check that the same potential is obtained for a one-parameter generalization of the Killing vector (2.44),

$$\mathcal{X} = V_I \left( \partial_{\tilde{\zeta}_I} - \zeta^I \partial_\sigma \right) + \kappa \partial_\sigma \tag{2.44}$$

In order to convince oneself that  $\partial_\sigma$  indeed the correct generator, recall that the symmetries which shift the scalars dual to the two Kaluza-Klein vector fields along  $\partial_\psi$  and  $\partial_t$  are  $E_k$  and  $E_{p_0}$ , respectively. Thus the gauging has to be a linear combination of these two generators. The only one which does not generate a potential is indeed  $\mathcal{X} = \kappa \partial_\sigma$ .

## 2.4 $\mathbb{R} \times \text{SU}(2) \times \text{U}(1)$ symmetric solutions

We now further specify to the case of stationary solutions with an  $\text{SU}(2) \times \text{U}(1)$  group of isometries. The metric on the three-dimensional slices can be taken to be

$$ds_3^2 = N^2(\rho) d\rho^2 + r^2(\rho) (\sigma_1^2 + \sigma_2^2), \quad (2.45)$$

where the lapse function  $N$  maintains reparameterization invariance along the radial direction  $\rho$ . Here,  $\sigma_i$  are the left-invariant  $\text{SU}(2)$  forms,<sup>4</sup>

$$\sigma_1 = \sin \psi d\theta - \cos \psi \sin \theta d\phi \quad (2.46)$$

$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi \quad (2.47)$$

$$\sigma_3 = d\psi + \cos \theta d\phi \quad (2.48)$$

$\theta, \phi, \psi$  are the Euler angles of  $S^3$  with ranges  $\theta \in [0, \pi)$ ,  $\phi \in [0, 2\pi)$ ,  $\psi \in [0, 4\pi)$ , such that

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk} \sigma^j \wedge \sigma^k \quad (2.49)$$

Moreover the coordinates  $\varphi^a$  on  $\mathcal{M}_3$  are taken to be functions of  $\rho$  only. It will be useful to relax the condition that the  $\psi$  circle has unit Chern class over the  $S^2$  parametrized by  $(\theta, \phi)$ , and define

$$\sigma_{3,k} = d\psi + ik \cos \theta d\phi \quad (2.50)$$

For  $k = -i$ , the Hopf fiber  $\text{U}(1)$  combines with  $S^2$  to produce the topology of  $S^3$ .

Upon reduction along the  $\theta$  and  $\phi$  direction, the Lagrangian

$$\mathcal{L}_1 = N^{-1} \left[ (r')^2 - r^2 g_{ab} \varphi'^a \varphi'^b \right] + N V_1 \quad (2.51)$$

describes, in a reparameterization invariant way, the motion of a fiducial particle on the cone  $\mathbb{R} \times \mathcal{M}_3$  in the presence of a potential

$$V_1 = 1 + r^2 e^{-2U} f^{-1} V_5. \quad (2.52)$$

In particular, the equation of motion of  $N$  enforces the Hamiltonian constraint

$$H_{\text{WDW}} \equiv \frac{N}{16} \left( p_r^2 - \frac{1}{r^2} g^{ab} p_{\varphi^a} p_{\varphi^b} - V_1 \right) = 0. \quad (2.53)$$

In the ungauged case, this mass of the particle is therefore fixed to unity, and the motion decouples between the cone direction and  $\mathcal{M}_3$ . In the gauged case, the mass is effectively position dependent, with a correction proportional to the radius  $re^{-U}/\sqrt{f}$  of the two-sphere measured in the five-dimensional metric. Moreover, the cone direction and  $\mathcal{M}_3$  no longer decouple. In either case, the phase space of  $\mathbb{R} \times \text{SU}(2) \times \text{U}(1)$  symmetric solutions of 5D supergravity is given by the symplectic quotient

$$T^*(\mathbb{R}^+ \times \mathcal{M}_3) // \{H_{\text{WDW}} = 0\}, \quad (2.54)$$

of dimension 16.

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<sup>4</sup>Note the exchange of  $\phi$  and  $\psi$  with respect to [7].

**Conserved charges and integrability.** Due to the isometries (2.23) of  $\mathcal{M}_3$ , there are many conserved quantities which can be used to integrate the motion. The conserved charges  $k, p^\Lambda, q_\Lambda, T_I$  are sufficient to eliminate the (derivatives of)  $\sigma, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \phi_I$ . Physically, the conserved quantity associated to  $T_I$  and  $q'_0$  correspond to the electric charge and angular momentum in 5 dimensions, whereas  $p^\Lambda, q_\Lambda$  are dipole-type charges. The charge  $k$  is the Chern class of the circle bundle generated by  $\partial_\psi$  over the two-sphere parameterized by  $(\theta, \phi)$ , and should be fixed to  $k = -i$  (after analytic continuation) in order that the total space have the topology of  $S^3$ . These identifications are to be contrasted with the ones relevant for describing four-dimensional black holes [23, 25], and are consistent with the “4D/5D lift” [4] as shown in section 3.1 below.

In the case where  $\mathcal{M}_3$  is a symmetric space  $G_3/K_3$  and in the absence of gauging, all solutions can in fact be obtained by exponentiating the action of the isometry group.<sup>5</sup> For this purpose, it is useful to parametrize  $G_3/K_3$  by an element  $g$  in the Iwasawa gauge, i.e. in the  $A_3N_3$  part of the Iwasawa decomposition of  $G_3 = K_3A_3N_3$  into the product of the maximal compact  $K_3$ , abelian torus  $A_3$  and nilpotent subgroup  $N_3$ . The right-invariant metric is obtained from the non-compact part  $p$  of the right-invariant one-form  $\theta = dg \cdot g^{-1}$  valued in the Lie algebra  $\mathfrak{g}_3$  of  $G_3$ ,

$$ds^2 = \text{Tr}(p^2), \quad \theta = h + p. \tag{2.55}$$

When  $G_3$  is represented by real matrices, the Cartan decomposition  $\theta = h + p$  is simply the decomposition into antisymmetric matrices  $h$  and symmetric matrices  $p$ . A geodesic passing through the point  $g_0$  at  $\tau = 0$  with initial velocity  $p_0$  is then given by  $g(\tau) = k(\tau) \cdot e^{p_0\tau} \cdot g_0$  where  $p_0$  is a non-compact element in  $\mathfrak{g}_3$ ,  $k(\tau)$  is the unique element of  $K_3$  which brings  $g(t)$  back to the Iwasawa gauge, and  $\tau$  is the affine parameter. The rotation  $k(\tau)$  drops from the product  $M(\tau) = g^t(\tau) \cdot g(\tau)$ , from which the coordinates on  $G_3/K_3$  can be read off. This produces a solution of the Lagrangian (2.51) in the gauge  $N = r^2$  (Conversely, given a solution of (2.51), the affine geodesic parameter  $\tau$  may be obtained by integrating  $N(\rho)d\rho/r^2(\rho) = -d\tau$ ). The remaining motion of  $r(\rho)$  may be obtained by integrating the Hamiltonian constraint (2.53), and depends only on  $p_0^2$ ; in particular, if  $(p^0)^2 = 0$ ,  $r = 1/(\tau + \gamma)$  where  $\gamma$  is an integration constant. The  $\mathfrak{g}_3$ -valued conserved charges inherited from the right action of  $G_3$  are then given by

$$Q = -dM M^{-1} = -g_0^t p_0 g_0^{-t} \tag{2.56}$$

Supersymmetric solutions correspond to special restrictions on the momentum  $p_0$ . In many cases, but not all,  $Q$  is nilpotent, i.e.  $Q^n = 0, Q^{n-1} \neq 0$  for some integer  $n$ . Again, in the presence of a potential  $V_3$ ,  $G_3$  is broken to a solvable subgroup and integrability is lost in general.

**Relation to Gauntlett et al. classification.** Supersymmetric solutions of  $D = 5, \mathcal{N} = 1$  supergravity were classified in [47] and [48] for the ungauged and gauged case, respectively.

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<sup>5</sup>This fact was used in [20] to produce explicit non-supersymmetric extremal solutions in  $D = 4, \mathcal{N} = 2$  very special supergravity with one modulus.

A necessary condition in the gauged case is that the four-dimensional metric  $ds_4^2$  in the square bracket of (2.57) has to be Kähler. In terms of the  $SU(2) \times U(1)$  symmetric ansatz used in [47, 48, 7], slightly generalized to include an arbitrary lapse function  $\mathcal{N}(\rho)$  and Chern class  $k$ ,

$$ds_5^2 = -f^2(dt + \omega_4)^2 + f^{-1} [\mathcal{N}^2 d\rho^2 + a^2(\sigma_1^2 + \sigma_2^2) + b^2 \sigma_{3,k}^2] \quad (2.57)$$

$$\omega_4 = \Psi \sigma_{3,k}, \quad A^I = f h^I (dt + \Psi \sigma_{3,k}) + U^I(\rho) \sigma_{3,k}, \quad (2.58)$$

a sufficient condition for Kählerity is given by [6]<sup>6</sup>

$$ikb\mathcal{N} - 2aa' = 0, \quad (2.59)$$

corresponding to a Kähler form  $\omega_K = -d(a^2 \sigma_{3,k})$ . In terms of the variables in our ansätze (2.16),(2.45), related to the ones above by

$$a = e^{-U} r, \quad b = e^U, \quad \mathcal{N} = N e^{-U}, \quad \Psi = 2i\zeta^0, \quad U^I = \sqrt{\frac{3}{2}} \zeta^I, \quad (2.60)$$

the condition (2.59) becomes

$$r' - rU' - ik \frac{N e^{2U}}{2r} = 0. \quad (2.61)$$

It should be stressed that conditions (2.59) or (2.61) are independent of the gauge coupling  $1/\ell$ . As we shall see below, some supersymmetric solutions of ungauged supergravity do not satisfy this condition (e.g. the Taub-NUT black hole). It therefore appears that more branches of solutions open up in the ungauged case  $\ell = \infty$ . It would be interesting to see if remnants of these branches exist at finite  $\ell$ .

### 3. Supersymmetric solutions in $D = 5$ , $\mathcal{N} = 1$ minimal supergravity

In this section, we specialize to the case of minimal  $\mathcal{N} = 1$  supergravity in 5 dimensions, possibly gauged, and restrict to stationary solutions with  $SU(2) \times U(1)$  group of isometries.

#### 3.1 Geometry of $G_{2(2)}/SO(4)$

After the three-step reduction process explained in the previous section, one obtains a one-dimensional Lagrangian

$$\mathcal{L} = \frac{1}{N} \left[ (r')^2 - r^2 (u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E}) \right] + N \left( 1 + \frac{6r^2 e^{-2U}}{\tau_2 \ell^2} \right) \quad (3.1)$$

corresponding to the motion of a particle on the symmetric space  $\mathcal{M}_3 = G_{2(2)}/Sl(2) \times Sl(2)$ , with metric

$$ds_{\mathcal{M}_3}^2 = J_{AB} \epsilon_{\dot{\alpha}\dot{\beta}} V^{\dot{\alpha}A} V^{\dot{\beta}B} = u \bar{u} + v \bar{v} + e \bar{e} + E \bar{E}, \quad (3.2)$$

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<sup>6</sup>This condition is in fact more restrictive than Kählerity. For example, it is obeyed for flat  $\mathbb{R}^4$  or Eguchi-Hanson, but not for Taub-NUT.

and, when  $\ell$  is finite, a position-dependent mass. Here,  $J^{AB}$  and  $\epsilon_{\dot{\alpha}\dot{\beta}}$  are antisymmetric forms in 2 and 4 dimensions, with the conventions  $\epsilon_{12} = J_{14} = J_{32} = 1$ , and  $V^{A\dot{\alpha}}$  is the quaternionic viel-bein with entries [21]

$$u = \frac{e^{-U}}{2\sqrt{2}\tau_2^{3/2}} \left( d\tilde{\zeta}_0 + \tau d\tilde{\zeta}_1 + 3\tau^2 d\zeta_1 - \tau^3 d\zeta_0 \right) \quad (3.3a)$$

$$v = dU - \frac{i}{2} e^{-2U} \left( d\sigma - \zeta_0 d\tilde{\zeta}_0 - \zeta_1 d\tilde{\zeta}_1 + \tilde{\zeta}_0 d\zeta_0 + \tilde{\zeta}_1 d\zeta_1 \right) \quad (3.3b)$$

$$e = -\frac{\sqrt{3}}{2\tau_2} d\tau \quad (3.3c)$$

$$E = -\frac{e^{-U}}{2\sqrt{6}\tau_2^{3/2}} \left( 3d\tilde{\zeta}_0 + d\tilde{\zeta}_1 (\tau + 2\bar{\tau}) + 3\bar{\tau} (2\tau + \bar{\tau}) d\zeta_1 - 3\bar{\tau}\tau^2 d\zeta_0 \right) \quad (3.3d)$$

The viel-bein  $V^{A\dot{\alpha}}$  corresponds to the projection of the right-invariant form  $\theta = dg \cdot g^{-1}$  on the non-compact part  $\mathfrak{p}$  of the Lie algebra of  $G_{2(2)}$ , parameterized in the Iwasawa gauge as in [21],

$$g = \tau_2^{-Y_0} \cdot e^{\sqrt{2}\tau_1 Y_+} \cdot e^{-UH} \cdot e^{-\zeta^0 E_{q_0} + \tilde{\zeta}_0 E_{p_0}} \cdot e^{-\sqrt{3}\zeta^1 E_{q_1} + \frac{\sqrt{3}}{3}\tilde{\zeta}_1 E_{p_1}} \cdot e^{\sigma E_k} \quad (3.4)$$

where  $\tau \equiv \phi^1 + if \equiv \tau_1 + i\tau_2$ . The entries of  $V^{A\dot{\alpha}}$  in (3.3) have been normalized in such a way that that  $S_-$  acts by left multiplication by the standard spin 3/2 raising operator  $\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$  on the matrix (2.27), while  $J_+$  acts by right multiplication with  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

The compact part of the right-invariant form  $\theta$  provides the  $SU(2) \times SU(2)$ -valued spin connection

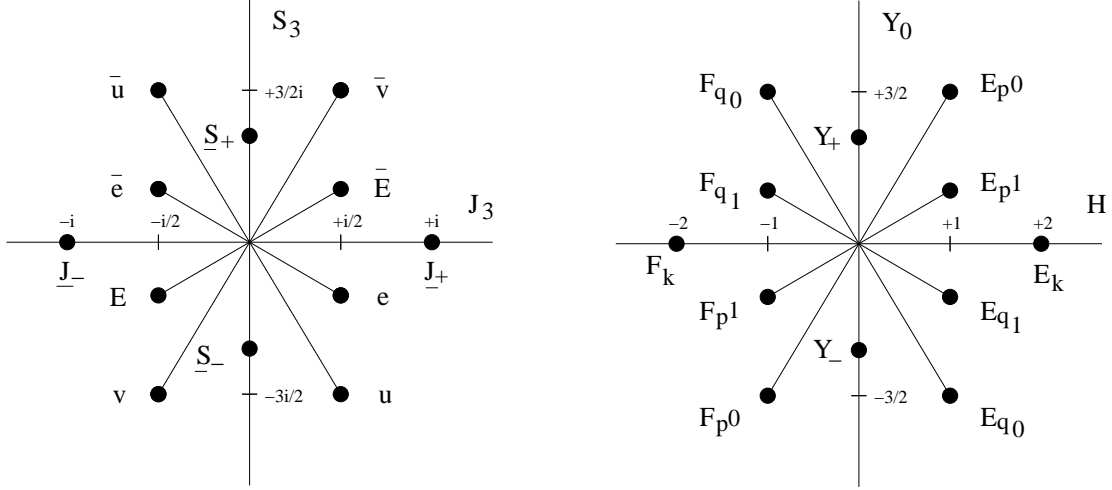
$$\begin{pmatrix} \underline{J}_+ \\ \underline{J}_3 \\ \underline{J}_- \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} u \\ \frac{i}{4}(v - \bar{v}) + \frac{\sqrt{3}}{4}(e + \bar{e}) \\ \bar{u} \end{pmatrix}, \quad \begin{pmatrix} \underline{S}_+ \\ \underline{S}_3 \\ \underline{S}_- \end{pmatrix} = \frac{\sqrt{3}}{2} \begin{pmatrix} E \\ -i\frac{\sqrt{3}}{4}(v - \bar{v}) + \frac{1}{4}(e + \bar{e}) \\ \bar{E} \end{pmatrix}. \quad (3.5)$$

As indicated below (2.20), we are taking  $\tau_1, \zeta^0, \tilde{\zeta}_1, \sigma$  to be purely imaginary, so that we can use the same expressions as in [21] which was taylored for the Riemannian space  $G_{2(2)}/SO(4)$ . Our notations for the components of the right-invariant form  $\theta$ , as well as for the Killing vectors of the right-action to be discussed presently, are summarized in figure 2.

**Conserved charges.** The metric (3.2) on the symmetric space  $G_{2(2)}/SO(4)$  is by construction invariant under the right-action of  $G_{2(2)}$  on the coset representative  $e$ , compensated by a left-action of its maximal compact subgroup such as to preserve the Iwasawa gauge (3.4). The corresponding Killing vectors were computed in [21]. Replacing the vector field  $\partial_{\phi^a}$  by the momentum  $p_{\phi^a}$  conjugate to  $\phi^a$ ,

$$p_U = 4\frac{r^2}{N}U', \quad p_{\tau_2} = 3\frac{r^2}{N\tau_2^2}\tau_2', \quad p_\sigma = \frac{r^2}{N}e^{-4U}(\sigma' + \tilde{\zeta}_I \zeta^{I'} - \zeta^I \tilde{\zeta}'_I), \dots \quad (3.6)$$





**Figure 2:** Left: components of the right-invariant form  $dg \cdot g^{-1}$ , in the basis appropriate to the compact Cartan torus  $J_3, S_3$ . Right: Killing vectors for the right-action of  $G_{2(2)}$ , in the basis appropriate to the split Cartan torus  $H, Y_0$ .

we find that the conserved charges associated to the right-action of  $G_{2(2)}$  are given by<sup>7</sup>

$$E_k = p_\sigma \tag{3.7a}$$

$$E_{p^0} = p_{\tilde{\zeta}_0} - \zeta^0 p_\sigma, \quad E_{q_0} = -p_{\zeta^0} - \tilde{\zeta}_0 p_\sigma \tag{3.7b}$$

$$E_{p^1} = \sqrt{3}(p_{\tilde{\zeta}_1} - \zeta^1 p_\sigma), \quad E_{q_1} = \frac{1}{\sqrt{3}}(-p_{\zeta^1} - \tilde{\zeta}_1 p_\sigma) \tag{3.7c}$$

$$H = -p_U - 2\sigma p_\sigma - \zeta^0 p_{\zeta^0} - \zeta^1 p_{\zeta^1} - \tilde{\zeta}_0 p_{\tilde{\zeta}_0} - \tilde{\zeta}_1 p_{\tilde{\zeta}_1} \tag{3.7d}$$

$$Y_+ = \frac{1}{\sqrt{2}}(p_{\tau_1} + \zeta^0 p_{\zeta^1} - 6\zeta^1 p_{\tilde{\zeta}_1} - \tilde{\zeta}_1 p_{\tilde{\zeta}_0}) \tag{3.7e}$$

$$Y_0 = -\frac{1}{2}(2\tau_1 p_{\tau_1} + 2\tau_2 p_{\tau_2} - 3\zeta^0 p_{\zeta^0} + 3\tilde{\zeta}_0 p_{\tilde{\zeta}_0} - \zeta^1 p_{\zeta^1} + \tilde{\zeta}_1 p_{\tilde{\zeta}_1}) \tag{3.7f}$$

$$Y_- = \frac{1}{3\sqrt{2}} \left( 6p_{\tau_2} \tau_1 \tau_2 + 3p_{\tau_1} (\tau_1^2 - \tau_2^2) + 9p_{\tilde{\zeta}_1} \tilde{\zeta}_0 - 9p_{\zeta^0} \zeta^1 + 2p_{\zeta^1} \tilde{\zeta}_1 \right) \tag{3.7g}$$

$$F_{q_0} = -\frac{1}{\tau_2^3} \left( 6p_{\tilde{\zeta}_1} \zeta^{1^2} - 2(p_{\tau_1} + p_{\zeta^1} \zeta^0) \zeta^1 - p_U \zeta^0 + 2\zeta^0 (p_{\tau_1} \tau_1 + p_{\tau_2} \tau_2 - p_{\zeta^0} \zeta^0) \right) \\ - p_{\tilde{\zeta}_0} (\sigma + \zeta^0 \tilde{\zeta}_0 + \zeta^1 \tilde{\zeta}_1) + p_\sigma \left( 2\zeta^{1^3} + \zeta^0 (-\sigma + \zeta^0 \tilde{\zeta}_0 + \zeta^1 \tilde{\zeta}_1) \right) \\ + e^{2U} \left( p_{\tilde{\zeta}_0} \tau_1^3 + p_\sigma \zeta^0 \tau_1^3 - 3p_{\tilde{\zeta}_1} \tau_1^2 - 3p_\sigma \zeta^1 \tau_1^2 + p_{\zeta^1} \tau_1 - p_\sigma \tilde{\zeta}_1 \tau_1 + p_{\zeta^0} - p_\sigma \tilde{\zeta}_0 \right) \tag{3.7h}$$

$$F_{q_1} = \frac{1}{3\sqrt{3}\tau_2^3} \tau_2^3 \left( 2(p_{\tilde{\zeta}_0} + p_\sigma \zeta^0) \tilde{\zeta}_1^2 + (-4p_{\tau_1} - 4p_{\zeta^1} \zeta^0 + 3\zeta^1 (5p_{\tilde{\zeta}_1} + p_\sigma \zeta^1)) \tilde{\zeta}_1 \right) \\ - 6(2p_{\tau_2} \tau_1 \tau_2 + p_{\tau_1} (\tau_1^2 - \tau_2^2)) \zeta^0 - 9p_{\tilde{\zeta}_1} (\sigma + \zeta^0 \tilde{\zeta}_0) + 9p_U \zeta^1$$

<sup>7</sup>For convenience, we stick to the notations in [21]. The generators  $p^\Lambda, q^\Lambda, k, H, D, T_I, S^I, p'_{\Lambda}, q'^{\Lambda}, k'$  introduced in (2.23) and subsequent equations in section 2 are equal to  $E_{p^\Lambda}, E_{q^\Lambda}, E_k, H, Y_0, Y_+, Y_-, F_{q^\Lambda}, F_{p^\Lambda}, F_k$ , respectively. The lowest root  $F_k$ , too bulky to be displayed here, can be obtained by Poisson commuting  $F_{p^I}$  and  $F_{q_I}$ .

$$\begin{aligned}
 & +3\zeta^1(-2p_{\tau_1}\tau_1 - 2p_{\tau_2}\tau_2 + 6p_{\zeta^0}\zeta^0 + 3p_{\sigma}(\sigma - \zeta^0\tilde{\zeta}_0) + 2p_{\zeta^1}\zeta^1) \\
 & - 3e^{2U} \left( 3p_{\zeta^0}\tau_1 + p_{\zeta^1}(3\tau_1^2 + \tau_2^2) + 3(p_{\tilde{\zeta}_0}\tau_1(\tau_1^2 + \tau_2^2) - p_{\tilde{\zeta}_1}(3\tau_1^2 + 2\tau_2^2) \right. \\
 & \quad \left. + p_{\sigma}(\zeta^0\tau_1^3 - 3\zeta^1\tau_1^2 + \tau_2^2\zeta^0\tau_1 - \tilde{\zeta}_0 - 2\tau_2^2\zeta^1) \right) \tau_1 - p_{\sigma}(3\tau_1^2 + \tau_2^2)\tilde{\zeta}_1 \quad (3.7i)
 \end{aligned}$$

$$\begin{aligned}
 F_{p^0} = & \frac{1}{27\tau_2^3} \left( 2p_{\sigma}\tilde{\zeta}_1^3 - 6p_{\zeta^1}\tilde{\zeta}_1^2 - 9(4p_{\tau_2}\tau_1\tau_2 + 2p_{\tau_1}(\tau_1^2 - \tau_2^2) + 6p_{\tilde{\zeta}_1}\tilde{\zeta}_0 - 3p_{\zeta^0}\zeta^1 + 3p_{\sigma}\tilde{\zeta}_0\zeta^1)\tilde{\zeta}_1 \right. \\
 & \left. - 27(p_{\zeta^0}(\sigma - \zeta^0\tilde{\zeta}_0) + \tilde{\zeta}_0(p_U + p_{\sigma}\sigma + 2p_{\tau_1}\tau_1 + 2p_{\tau_2}\tau_2 + 2p_{\tilde{\zeta}_0}\tilde{\zeta}_0 + p_{\sigma}\zeta^0\tilde{\zeta}_0)) \right) \tau_2^3 \\
 & + 27e^{2U} \left( p_{\tilde{\zeta}_0}\tau_1^6 + p_{\sigma}\zeta^0\tau_1^6 - 3p_{\tilde{\zeta}_1}\tau_1^5 - 3p_{\sigma}\zeta^1\tau_1^5 + 3p_{\tilde{\zeta}_0}\tau_2^2\tau_1^4 + 3p_{\sigma}\tau_2^2\zeta^0\tau_1^4 - 6p_{\tilde{\zeta}_1}\tau_2^2\tau_1^3 \right. \\
 & \quad + p_{\zeta^0}\tau_1^3 - p_{\sigma}\tilde{\zeta}_0\tau_1^3 - 6p_{\sigma}\tau_2^2\zeta^1\tau_1^3 + 3p_{\tilde{\zeta}_0}\tau_2^4\tau_1^2 + p_{\zeta^1}(\tau_1^2 + \tau_2^2)\tau_1^2 + 3p_{\sigma}\tau_2^4\zeta^0\tau_1^2 \\
 & \quad \left. - p_{\sigma}(\tau_1^2 + \tau_2^2)\tilde{\zeta}_1\tau_1^2 - 3p_{\tilde{\zeta}_1}\tau_2^4\tau_1 - 3p_{\sigma}\tau_2^4\zeta^1\tau_1 + p_{\tilde{\zeta}_0}\tau_2^6 + p_{\sigma}\tau_2^6\zeta^0 \right) \quad (3.7j)
 \end{aligned}$$

$$\begin{aligned}
 F_{p^1} = & \frac{1}{3\sqrt{3}\tau_2^3} \left( - \left( (2p_{\tilde{\zeta}_1} - p_{\sigma}\zeta^1)\tilde{\zeta}_1^2 + 2p_{\tau_1}\tau_1\tilde{\zeta}_1 + (3p_U + 2p_{\tau_2}\tau_2 + 6p_{\tilde{\zeta}_0}\tilde{\zeta}_0 + 3p_{\sigma}(\sigma + \zeta^0\tilde{\zeta}_0))\tilde{\zeta}_1 \right. \right. \\
 & \quad - 6p_{\tau_1} \left( -2\zeta^1\tau_1^2 + \tilde{\zeta}_0 + 2\tau_2^2\zeta^1 \right) + 6\zeta^1(4p_{\tau_2}\tau_1\tau_2 + 6p_{\tilde{\zeta}_1}\tilde{\zeta}_0 - 3p_{\zeta^0}\zeta^1 + 3p_{\sigma}\tilde{\zeta}_0\zeta^1) \\
 & \quad + p_{\zeta^1}(3\sigma - 3\zeta^0\tilde{\zeta}_0 + 5\zeta^1\tilde{\zeta}_1) \left. \right) \tau_2^3 - 3e^{2U} \left( 3p_{\zeta^0}\tau_1^2 - p_{\sigma}(3\tau_1^2 + 2\tau_2^2)\tilde{\zeta}_1\tau_1 \right. \\
 & \quad + p_{\zeta^1}(3\tau_1^3 + 2\tau_2^2\tau_1) + 3 \left( -p_{\tilde{\zeta}_1}(\tau_1^2 + \tau_2^2)(3\tau_1^2 + \tau_2^2) - p_{\sigma}(\tau_1^2 + \tau_2^2)\zeta^1(3\tau_1^2 + \tau_2^2) \right. \\
 & \quad \left. \left. + \tau_1 \left( (\tau_1^2 + \tau_2^2)^2 (p_{\tilde{\zeta}_0} + p_{\sigma}\zeta^0) - p_{\sigma}\tau_1\tilde{\zeta}_0 \right) \right) \right) \quad (3.7k)
 \end{aligned}$$

As we shall see presently, the conserved charges  $Y_+$  and  $F_{q^0}$  correspond to the electric charge and angular momentum, respectively.

For the most part, we will be interested in purely electric solutions, which satisfy  $E_{p^0} = E_{p^1} = 0$ ,  $E_k = k$  and  $Y_+ = q$ . In this case, we may solve for the time derivatives of the corresponding scalars as follows:

$$\tau_1' = -2e^{-2U}\tau_2^3\zeta^0(\zeta^{1'} - \zeta^{0'}\tau_1) + 2\frac{kN\tau_2^2}{r^2}(\zeta^1 - \tau_1\zeta^0)^2 + \frac{\sqrt{2}qN\tau_2^2}{3r^2}, \quad (3.8)$$

$$\tilde{\zeta}_0' = -2\zeta^{0'}\tau_1^3 + 3\zeta^{1'}\tau_1^2 + e^{2U}\frac{kN}{r^2}\tau_2(3\tau_1^2\zeta^0 - 3\tau_1\zeta^1 + \tau_2^2\zeta^0) \quad (3.9)$$

$$\tilde{\zeta}_1' = 3\tau_1^2\zeta^{0'} - 6\tau_1\zeta^{1'} + 3e^{2U}\frac{kN}{r^2}\tau_2(\zeta^1 - \tau_1\zeta^0) \quad (3.10)$$

$$\sigma' = e^{4U}\frac{kN}{r^2} + \zeta^0\tilde{\zeta}_0' + \zeta^1\tilde{\zeta}_1' - \tilde{\zeta}_0\zeta^{0'} - \tilde{\zeta}_1\zeta^{1'} \quad (3.11)$$

Moreover, we focus mostly on solutions with  $k = -i$ , such that the angular directions  $(\theta, \phi, \psi)$  parametrize a topological  $S^3$ .

**4D-5D lift and  $(t, \psi)$  flip.** It is also useful to introduce a different parametrization, adapted to the  $Sl(2, \mathbb{R})$  subgroup corresponding to the diffeomorphisms of the  $(t, \psi)$  torus:

$$g = V^{-\frac{1}{4}(3H+2Y_0)} \cdot \rho_2^{-\frac{1}{4}(H-2Y_0)} \cdot e^{\frac{\sqrt{2}}{2}E_{q_0}\rho_1} \cdot e^{\mu_1 E_{q_1} + \frac{2}{\sqrt{2}}\mu_2 Y_+} \cdot e^{\nu E_{p^1}} \cdot e^{\sqrt{2}\tilde{\mu}_1 E_k + \tilde{\mu}_2 E_{p^0}} \quad (3.12)$$

The variables  $(V, \rho_1, \rho_2, \mu_1, \mu_2, \nu, \tilde{\mu}_1, \tilde{\mu}_2)$  are related to the previous ones by

$$V = e^U \sqrt{\tau_2}, \quad \rho_1 = -\sqrt{2}\zeta^0, \quad \rho_2 = \frac{e^U}{\tau_2^{3/2}} \quad \mu_1 = \sqrt{3}(\tau_1\zeta^0 - \zeta^1), \quad \mu_2 = \sqrt{\frac{3}{2}}\tau_1, \quad (3.13)$$

$$\nu = \frac{3\tau_1\zeta^1 + \tilde{\zeta}_1}{\sqrt{3}}, \quad \tilde{\mu}_1 = \frac{\sigma + \zeta^0(\tilde{\zeta}_0 - 2\tau_1^2\zeta^1) + \zeta^1(4\tau_1\zeta^1 + \tilde{\zeta}_1)}{\sqrt{2}} \quad \tilde{\mu}_2 = \tilde{\zeta}_0 - \tau_1^2\zeta^1, \quad (3.14)$$

The parameter  $\rho = \rho_1 + i\rho_2$  transforms under this  $Sl(2)$  by fractional linear transformations, while  $(\mu_1, \mu_2)$  and  $(\tilde{\mu}_1, \tilde{\mu}_2)$  transform as doublets, and  $\nu$  is inert. In particular, under the Weyl reflection

$$\rho \rightarrow -1/\rho, \quad (\mu_1, \mu_2) \rightarrow (\mu_2, -\mu_1), \quad \nu \rightarrow \nu, \quad (\tilde{\mu}_1, \tilde{\mu}_2) \rightarrow (\tilde{\mu}_2, -\tilde{\mu}_1) \quad (3.15)$$

the coordinates  $U, \tau, \zeta^I, \tilde{\zeta}_I$  transform into

$$e^U \rightarrow \frac{e^U \tau_2^{3/4}}{(2\zeta^{02}\tau_2^3 + e^{2U})^{1/4}}, \quad \tau_1 \rightarrow \sqrt{2}(\zeta^1 - \tau_1\zeta^0), \quad \tau_2 \rightarrow \sqrt{\frac{2\zeta^{02}\tau_2^3 + e^{2U}}{\tau_2}}, \quad (3.16)$$

$$\zeta^0 \rightarrow -\frac{\zeta^0\tau_2^3}{2\zeta^{02}\tau_2^3 + e^{2U}}, \quad \zeta^1 \rightarrow -\frac{2\zeta^0\zeta^1\tau_2^3 + e^{2U}\tau_1}{\sqrt{2}(2\zeta^{02}\tau_2^3 + e^{2U})}, \quad (3.17)$$

$$\tilde{\zeta}_0 \rightarrow \frac{1}{\sqrt{2}} \left( -2\zeta^{02}\tau_1^3 + 6\zeta^0\zeta^1\tau_1^2 - 6\zeta^{12}\tau_1 + \frac{4\tau_2^3\zeta^0(\tau_1\zeta^0 - \zeta^1)^3}{2\zeta^{02}\tau_2^3 + e^{2U}} - \sigma - \zeta^0\tilde{\zeta}_0 - \zeta^1\tilde{\zeta}_1 \right), \quad (3.18)$$

$$\tilde{\zeta}_1 \rightarrow \frac{6\zeta^0(\zeta^1 - \tau_1\zeta^0)^2\tau_2^3}{2\zeta^{02}\tau_2^3 + e^{2U}} - 3\tau_1^2\zeta^0 + 6\tau_1\zeta^1 + \tilde{\zeta}_1, \quad (3.19)$$

$$\sigma \rightarrow \frac{\zeta^0(-\sigma + 3\zeta^0\tilde{\zeta}_0 + \zeta^1\tilde{\zeta}_1)\tau_2^3 + e^{2U}(\zeta^0\tau_1^3 + \tilde{\zeta}_1\tau_1 + 2\tilde{\zeta}_0)}{\sqrt{2}(2\zeta^{02}\tau_2^3 + e^{2U})}, \quad (3.20)$$

Note that this transformation maps the reality conditions (2.21) appropriate to 5D black holes, to the conditions to (2.22) appropriate to 4D black holes. As we shall see on an example in the next section, it implements the 4D/5D lift found in [4].

**Poisson algebra of the viel-bein components.** In order to describe the constraints from unbroken supersymmetry, it will be useful to compute the Poisson brackets of the entries in the quaternionic viel-bein.<sup>8</sup> By this we mean the following: the entries in  $V^{A\dot{\alpha}}$  are one-forms on  $\mathcal{M}_3$ , which may be pulled back to the world-line into one-forms  $v_a(\varphi)\varphi^{a'}d\tau$ ; expressing the velocities  $\varphi^{a'}$  in terms of the momenta  $p_a$ , we obtain functions on the phase space  $T^*(\mathbb{R}^+ \times \mathcal{M}_3)$ , whose Poisson bracket can be computed in the usual way. Equivalently, the one-form  $v_a(\varphi)$  may be turned into a vector field  $\mathcal{X}^a(\varphi)$  using the metric on  $\mathbb{R}^+ \times \mathcal{M}_3$ , and the Poisson bracket of  $V$  is just the Lie bracket of  $\mathcal{X}$ . The result of this computation is that the non-vanishing Poisson brackets among the entries of  $V^{A\dot{\alpha}}$ , up to complex conjugation, are given by

$$\{u, \bar{u}\} = \{v, \bar{v}\} = \{E, \bar{E}\} = \frac{N}{2r^2}(\bar{v} - v), \quad \{e, \bar{e}\} = -\frac{iN}{2\sqrt{3}r^2}(e + \bar{e}) \quad (3.21a)$$

<sup>8</sup>Some of the results in this subsection were obtained in collaboration with A. Waldron in 2006.

$$\{u, v\} = \{u, \bar{v}\} = -\frac{N}{4r^2}u, \quad \{E, v\} = \{E, \bar{v}\} = -\frac{N}{4r^2}E \quad (3.21b)$$

$$\{u, e\} = -i\frac{N\sqrt{3}}{4r^2}u, \quad \{u, \bar{e}\} = \frac{iN}{2r^2}\left(\bar{E} - \frac{\sqrt{3}}{2}u\right) \quad (3.21c)$$

$$\{E, e\} = \frac{iN}{4\sqrt{3}r^2}(E + 4\bar{E}), \quad \{E, \bar{e}\} = \frac{iN}{2r^2}\left(\bar{u} + \frac{\sqrt{3}}{6}E\right) \quad (3.21d)$$

A useful observation is that, up to terms proportional to  $V$  and  $V'$ , the Poisson bracket  $\{V, V'\}$  is given by a linear combination of the spin connections  $\underline{J}_i$  and  $\underline{S}_i$  in (3.5), with coefficients given by the Clebsh-Gordan coefficients for the tensor product  $(2, 4) \wedge (2, 4) = (1, 3) + (3, 1) + (1, 7) + (3, 5)$  in  $SU(2) \times SU(2)$ . In particular, the conservation of the  $J_3$  and  $S_3$  charges may be checked easily from figure (2) (left). The commutation relations (3.21) can be summarized in the following formula

$$\{V^{\dot{\alpha}A}, V^{\dot{\beta}B}\} = \frac{2i}{3}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma_4^i)^A J^{CB} \underline{S}_i + 2iJ^{AB}(\sigma^i)^{\dot{\alpha}} \epsilon^{\dot{\gamma}\dot{\beta}} \underline{J}_i + \dots \quad (3.22)$$

where  $J^{AB}$  and  $\epsilon_{\dot{\alpha}\dot{\beta}}$  are the antisymmetric forms in 2 and 4 dimensions with the conventions  $\epsilon_{12} = J_{14} = J_{32} = 1$ . The dots in this expression denote the  $(1, 7)$  and  $(3, 5)$  pieces in the tensor product, which vanish when either  $V^{\dot{\alpha}A}$  or  $V^{\dot{\beta}B}$  vanish.<sup>9</sup>

Using (3.21), it is straightforward to compute the commutation relations of the Hamiltonian  $H_0 = u\bar{u} + v\bar{v} + e\bar{e} + E\bar{E}$  with the viel-bein:

$$\{H_0, u\} = \frac{iN}{4r^2}\left(u\left(\sqrt{3}e + \sqrt{3}\bar{e} - 3iv + i\bar{v}\right) - 2e\bar{E}\right) \quad (3.23a)$$

$$\{H_0, v\} = -\frac{N}{2r^2}(E\bar{E} + u\bar{u} + v(\bar{v} - v)) \quad (3.23b)$$

$$\{H_0, e\} = \frac{iN}{2\sqrt{3}r^2}\left(e(e + \bar{e}) + 2\bar{E}^2 - 2\sqrt{3}Eu\right) \quad (3.23c)$$

$$\{H_0, E\} = \frac{N}{4\sqrt{3}r^2}\left(-iE\bar{e} - 4i\bar{e}\bar{E} - ie\left(E + 2\sqrt{3}\bar{u}\right) + \sqrt{3}E(3v - \bar{v})\right) \quad (3.23d)$$

These relations will become useful when analyzing the algebra of constraints imposed by supersymmetry in the next section.

**Shifted quaternionic vielbein.** In view of the supersymmetry variation (2.39) in the gauged case, we define the “shifted quaternionic vielbein”

$$\underline{V}^{A\dot{\alpha}} = V_a^{A\dot{\alpha}}\left(d\varphi^a - \frac{3i\sqrt{2}}{\ell}N\mathcal{X}^a\right), \quad (3.24)$$

where  $\mathcal{X} = \partial_{\zeta_1} - \zeta^1\partial_\sigma$  is the Killing vector controlling the gauging, and the normalization has been chosen to agree with the analysis in [6]. In terms of the entries of the “shifted

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<sup>9</sup>A more conceptual interpretation of the Poisson algebra (3.21) is that it is isomorphic to the Borel subalgebra of  $G_{2(2)}$ , after an appropriate change of basis.

vielbein",

$$\begin{aligned} \tilde{u} &= u + \frac{3e^{-U}N}{2\ell\tau_2^{3/2}}(\tau_2 - i\tau_1), & \tilde{v} &= v + \frac{3\sqrt{2}}{\ell}Ne^{-2U}\zeta^1, \\ \tilde{E} &= E + \frac{\sqrt{3}e^{-U}N}{2\ell\tau_2^{3/2}}(\tau_2 + 3i\tau_1), & \tilde{e} &= e \end{aligned} \quad (3.25)$$

Note that the deformation does not commute with complex conjugation:

$$\begin{aligned} \bar{\tilde{u}} &= \bar{u} - \frac{3e^{-U}N}{2\ell\tau_2^{3/2}}(\tau_2 + i\tau_1), & \bar{\tilde{v}} &= \bar{v} - \frac{3\sqrt{2}}{\ell}Ne^{-2U}\zeta^1, \\ \bar{\tilde{E}} &= \bar{E} - \frac{\sqrt{3}e^{-U}N}{2\ell\tau_2^{3/2}}(\tau_2 - 3i\tau_1), & \bar{\tilde{e}} &= \bar{e} \end{aligned}$$

### 3.2 The BMPV and Taub-NUT black holes

In the next three sections, we various reductions of the one-dimensional dynamical system relevant for different kind of black holes.

**Constraint analysis.** In the absence of gauging, the one-dimensional system is the same as the one which arises in the study of four-dimensional black holes [23–25]. In this case, the supersymmetry conditions are given by

$$r' = N \quad \text{and} \quad \exists \epsilon_{\dot{\alpha}} \quad \text{such that} \quad V^{A\dot{\alpha}}\epsilon_{\dot{\alpha}} = 0. \quad (3.26)$$

The second condition is equivalent to the vanishing of the  $2 \times 2$  minors of the quaternionic vielbein,

$$H_{AB} \equiv V^{A\dot{\alpha}}V^{B\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} = 0. \quad (3.27)$$

In particular, it implies the vanishing of the total momentum  $P^2 = H_{AB}\Sigma^{AB}$  on  $\mathcal{M}_3$ , and together with  $r' = N$  the vanishing of the total Hamiltonian (2.53).

Using (3.21), one may check that the Poisson bracket of  $H_{\text{WDW}}$  with the constraints  $H_{AB}$  vanishes on the locus  $H_{AB}$ , for example

$$\{H_{12}, H_{13}\} = -\frac{\sqrt{3}}{3}uH_{12} + \frac{\sqrt{3}}{6}vH_{13} + \frac{i}{2}eH_{13} = 0. \quad (3.28)$$

This implies that the projectivized Killing spinor  $z = \epsilon_2/\epsilon_1$  can be computed consistently from any of the equations  $z = -V^{A\dot{1}}/V^{A\dot{2}}$ ; using again (3.21), one may check that

$$z' - \frac{1}{2} \left[ \bar{u} - \frac{1}{2}[(v - \bar{v}) - i\sqrt{3}(e + \bar{e})]z + z^2u \right] = 0. \quad (3.29)$$

Using the results in [21], one may check that this is precisely the condition  $(dz + P)/d\rho = 0$ , where  $P$  is the projectivized  $\text{SU}(2)$  connection on the quaternionic-Kähler manifold  $\mathcal{M}_3$ . Therefore, the motion can be lifted to a holomorphic geodesic on the twistor space  $Z = G_{2(2)}/\text{SU}(2) \times \text{U}(1)$  [25, 24].

**The generalized BMPV black hole.** It is straightforward to check that the constraint (3.26) indeed describes the supersymmetric spinning black hole constructed in [3]. In fact, the BMPV black hole is part of a family of solutions given by [47]

$$\mathcal{N} = 1, \quad a = b = \frac{\rho}{2}, \quad \tau_2 = -i\tau_1 = f = \left( \lambda + \frac{\mu}{\rho^2} + \frac{\chi^2}{9}\rho^2 \right)^{-1}, \quad (3.30)$$

$$\Psi = \frac{j}{2\rho^2} + \frac{\chi\mu}{2} + \frac{\chi\lambda}{4}\rho^2 + \frac{\chi^3}{54}\rho^4, \quad U^1 = \frac{1}{4\sqrt{3}}\chi\rho^2, \quad (3.31)$$

where  $\mu$  is the electric charge, equal to the ADM mass by the BPS condition,  $j$  is the angular momentum, and  $\chi$  is a deformation parameter which does not preserve the asymptotic flatness. Using (3.8), we easily obtain the remaining coordinates of the non-linear sigma model,

$$\sigma = -i\frac{\rho^2}{4} + i\frac{j\tilde{\zeta}_0}{2\sqrt{2}\rho^2} + i\frac{\lambda\chi\tilde{\zeta}_0\rho^2}{4\sqrt{2}} - \frac{\chi\tilde{\zeta}_1\rho^2}{6\sqrt{2}} + i\frac{\chi^3\rho^4\tilde{\zeta}_0}{54\sqrt{2}} \quad (3.32)$$

while  $\tilde{\zeta}_0$  and  $\tilde{\zeta}_1$  take constant values. It is easy to check that the first column<sup>10</sup> of the viel-bein  $V$  vanishes:<sup>11</sup>

$$\bar{u} = \bar{e} = E = v = 0 \quad (3.33)$$

so that the supersymmetry constraints (3.26) are obeyed at  $z = 0$ , which is indeed a fixed point of (3.29) when  $\bar{u} = 0$ . It would be interesting to find a more general class of solutions where an arbitrary linear combination of the two columns of the quaternionic viel-bein vanishes. At any rate, the fact that the solution preserves the same supersymmetry condition as the one appropriate for 4D black holes is consistent with the fact that the Killing spinor preserved by the 5D solution is independent of the  $\psi$  direction [49].

It is also instructive to compute the conserved charges (3.7) for the generalized BMPV solution:

$$E_{p^0} = E_{p^1} = 0, \quad E_{q_0} = 2i\tilde{\zeta}_0, \quad E_{q_1} = \frac{2i}{\sqrt{3}}\tilde{\zeta}_1, \quad E_k = -i, \quad (3.34)$$

$$Y_0 = 0, \quad Y_+ = \frac{3i\mu}{4\sqrt{2}}, \quad Y_- = -\frac{i\sqrt{2}\tilde{\zeta}_1^2}{3}, \quad H = \frac{\mu\chi\tilde{\zeta}_0}{\sqrt{2}}, \quad (3.35)$$

$$F_{p^0} = \frac{\chi\mu\tilde{\zeta}_0^2}{\sqrt{2}} + \frac{4i\tilde{\zeta}_1^3}{27}, \quad F_{p^1} = \frac{\mu\tilde{\zeta}_0(3i + \sqrt{2}\chi\tilde{\zeta}_1)}{2\sqrt{3}}, \quad F_{q_1} = -\frac{i\mu\tilde{\zeta}_1}{\sqrt{3}}, \quad F_{q_0} = -\frac{ij}{2\sqrt{2}}, \quad (3.36)$$

$$F_k = -\frac{i}{24} \left[ 6\sqrt{2}j\tilde{\zeta}_0 + \mu(3\chi^2\mu\tilde{\zeta}_0^2 + 4\tilde{\zeta}_1^2) \right], \quad (3.37)$$

This confirms the identification of  $Y_+$  and  $F_{q_0}$  as the electric charge and angular momentum, respectively. Moreover, one may check that the full Noether charge (viewed as a  $7 \times 7$  matrix via  $G_{2(2)} \subset \text{SO}(3,4)$ , see [21]) is nilpotent of degree 3,  $Q^3 = 0$ . This is a general consequence of the supersymmetry condition (3.26) in the symmetric space

<sup>10</sup>This could be traded for the second column upon flipping the sign of  $\tau_1, \Psi, \tilde{\zeta}_1, \sigma$ , corresponding to a flip of  $i$  in (2.21).

<sup>11</sup>Moreover, we observe that  $\bar{E} = 0$  when  $\chi = 0$ .

case [23]. However, it follows more generally from requiring extremality: indeed, a smooth near-horizon geometry is obtained provided  $a^2/f$  and  $b^2/f$  take a finite value as  $\rho \rightarrow 0$ , and  $f \sim \rho^2$  in the gauge  $\mathcal{N} = 1$ . In terms of the affine geodesic parameter  $\tau = 4/\rho^2$ , we see that the entries in the matrix  $M = e(\tau)e^t(\tau)$  (particularly the entry  $M_{44} = e^{-2U}/\tau_2 = 1/(b^2 f)$ , in the basis used in [21]) grow at most like  $\tau^2$ , consistently with  $Q^3 = 0$ .

**The Taub-NUT black hole.** Another solution in the same category is the rotating BPS black hole at the tip of a Taub-NUT space [4]<sup>12</sup> with NUT charge  $L$ , electric charge  $\mu$  and angular momentum  $j$

$$\mathcal{N} = \sqrt{1 + \frac{L}{\rho}}, \quad a = \rho \sqrt{1 + \frac{L}{\rho}}, \quad b = 1/\sqrt{1 + \frac{L}{\rho}}, \quad \tau_2 = -i\tau_1 = f = \left(1 + \frac{\mu}{\rho}\right)^{-1}, \quad (3.38)$$

$$\Psi = \frac{j}{L} \left(1 + \frac{L}{\rho}\right), \quad \tilde{\zeta}_0 = \tilde{\zeta}_1 = 0, \quad \sigma = \frac{iL}{\rho + L}. \quad (3.39)$$

The only non-vanishing conserved charges are given by

$$E_k = iL, \quad H = -2L, \quad Y_+ = \frac{3i\mu}{\sqrt{2}}, \quad F_{q_0} = -2ij\sqrt{2}, \quad F_k = iL \quad (3.40)$$

while the affine geodesic parameter is  $\tau = 1/\rho$ . Just like the BMPV black hole (3.30), the conditions (3.33) are obeyed and  $Q^3 = 0$  (as the two share the same near horizon geometry). In fact the two families of solutions are related by a ‘‘D-transformation’’ in the language of [40, 41], i.e. a Weyl reflection  $D = \exp\left[\frac{i\pi}{4}(E_k + F_k)\right]$ . Moreover, the solution (3.38) is related by a  $(t, \psi)$  flip (3.16) to a 4D BPS black hole with Noether charges

$$E_{p^0} = -iL\sqrt{2}, \quad E_{p^1} = 0, \quad E_{q_1} = -i\mu\sqrt{6}, \quad E_{q_0} = 2ij\sqrt{2}, \quad E_k = 0. \quad (3.41)$$

This is consistent with the 4D/5D lift [4]. For the pure Taub-NUT vacuum, i.e. when  $j = \mu = 0$ , we note that the matrix of Noether charges becomes nilpotent of degree 2. In appendix A, we provide more general solutions of this type.

**The semi-classical radial wave function of BMPV-type black holes.** The five constraints

$$\bar{u} = \bar{e} = E = v = p_r + 4 = 0 \quad (3.42)$$

are solved in the sector

$$Y_+ = q, \quad E_k = k \quad (3.43)$$

by setting  $p_a = \partial_{\phi^a} S$ , where

$$S_{q,k,f} = -4r + ike^{2U} + \sqrt{2}q\bar{\tau} + k \left[ \sigma + \zeta^0 \tilde{\zeta}_0 + \zeta^1 \tilde{\zeta}_1 + 6\bar{\tau}\zeta_1^2 - 6\bar{\tau}^2\zeta^0\zeta^1 + 2\bar{\tau}^3(\zeta^0)^2 \right] \\ + f \left( \tilde{\zeta}_0 + \bar{\tau}\tilde{\zeta}_1 + 3\bar{\tau}^2\zeta^1 - \bar{\tau}^3\zeta^0, \tilde{\zeta}_1 + 6\bar{\tau}\zeta^1 - 3\bar{\tau}^2\zeta^0 \right) \quad (3.44)$$

---

<sup>12</sup>This may be obtained as a special case  $\lambda = 1, \chi = 0, \delta = \mu/\sqrt{L}, a = \sqrt{L/4}$  of the family of rotating Taub-NUT solutions in [47] (eq. 3.57-59), upon changing coordinates from  $\rho$  to  $r$  such that  $\rho = 2a(r - a)$ .

and  $f$  is an arbitrary function of two variables. Thus, a basis of solutions of the quantum constraints in the semi-classical approximation is given by

$$\Psi_{q,k,p^0,p^1} \sim \exp \left[ iS_{q,k,0} + ip^0 \left( \tilde{\zeta}_0 + \bar{\tau}\tilde{\zeta}_1 + 3\bar{\tau}^2\zeta^1 - \bar{\tau}^3\zeta^0 \right) + ip^1 \left( \tilde{\zeta}_1 + 6\bar{\tau}\zeta^1 - 3\bar{\tau}^2\zeta^0 \right) \right] \quad (3.45)$$

While  $p^0$  is indeed the eigenvalue of  $E_{p^0}$ ,  $E_{p^1}$  and  $F_{q_0}$  do not commute with  $Y_+$ , and so cannot be diagonalized simultaneously. Note that the wave function is a anti-holomorphic function of  $\tau$ , as required by the condition  $\bar{e} = 0$ , and that it flattens out in the near horizon region where  $a, b \rightarrow 0$ , in the case of 4D black holes [24, 25].

### 3.3 The Gödel and Eguchi-Hanson black holes

We now turn to a second class of supersymmetric solutions, which turn out to satisfy a different set of constraints.

**The Gödel black hole.** In addition to the above families of solutions in  $D = 5$  minimal (ungauged) supergravity, a second class of supersymmetric solutions with isometries  $SU(2) \times U(1)$  was constructed [47]:

$$\mathcal{N} = 1, \quad a = b = \frac{\rho}{2}, \quad \tau_2 = i\tau_1 = f = \left( \lambda + \frac{\mu}{\rho^2} + \frac{\chi^2}{27\rho^6} \right)^{-1}, \quad (3.46a)$$

$$U^1 = \frac{\chi}{4\sqrt{3}\rho^2}, \quad \Psi = \gamma\rho^2 - \chi \left( \frac{\lambda}{4\rho^2} + \frac{\mu}{6\rho^4} + \frac{\chi^2}{270\rho^8} \right) \quad (3.46b)$$

Note that the relative sign between  $\tau_1$  and  $i\tau_2$  is flipped with respect to the BMPV solution (3.30). As above,  $\tilde{\zeta}_0$  and  $\tilde{\zeta}_1$  take constant values, while

$$\sigma = -\frac{i}{4} \left( 1 - 2\sqrt{2}\gamma\tilde{\zeta}_0 \right) \rho^2 - \frac{\chi(2\tilde{\zeta}_1 + 3i\lambda\tilde{\zeta}_0)}{12\sqrt{2}\rho^2} - \frac{i\chi\mu\tilde{\zeta}_0}{6\sqrt{2}\rho^4} - \frac{i\chi^3\tilde{\zeta}_0}{270\sqrt{2}\rho^8} \quad (3.46c)$$

These solutions are deformations of the generalized Gödel solution obtained by setting  $\mu = \chi = 0$ . For these Gödel-type solutions, the conserved charges are given by

$$E_{p^1} = E_{p^0} = 0, \quad E_{q_0} = 2i\tilde{\zeta}_0, \quad E_{q_1} = \frac{2i}{\sqrt{3}}\tilde{\zeta}_1, \quad E_k = -i, \quad (3.47a)$$

$$H = Y_0 = 0, \quad Y_+ = -\frac{3i\mu}{4\sqrt{2}}, \quad Y_- = -\frac{i\sqrt{2}\tilde{\zeta}_1^2}{3}, \quad (3.47b)$$

$$F_{q_0} = \frac{i\chi\lambda}{4\sqrt{2}}, \quad F_{q_1} = \frac{1}{4\sqrt{3}} \left( \sqrt{2}\chi + 4i\mu\tilde{\zeta}_1 \right), \quad F_{p^1} = -\frac{i}{2}\sqrt{3}\mu\tilde{\zeta}_0, \quad F_{p^0} = \frac{4i\tilde{\zeta}_1^3}{27}, \quad (3.47c)$$

$$F_k = \frac{i}{6}\mu\tilde{\zeta}_1^2 + \frac{\sqrt{2}}{24}\chi \left( 2\tilde{\zeta}_1 + 3i\lambda\tilde{\zeta}_0 \right) \quad (3.47d)$$

and the geodesic affine parameter is  $\tau = 4/\rho^2$ . Unlike the BMPV and Taub-NUT black holes, which had  $Q^3 = 0$ , one may check that the Noether charge matrix is now nilpotent of degree 7,  $Q^7 = 0$ . This shows that these solutions do not arise from the same supersymmetry constraint (3.26). Indeed, one may check that three out of the four entries of the *second column* of the quaternionic viel-bein vanishes,

$$e = \bar{E} = u = 0, \quad (3.48)$$



moreover

$$C_1 \equiv ik\bar{v} - N e^{-2U} = 0, \quad C_2 \equiv r' - rU' - ik \frac{N e^{2U}}{2r} = 0 \quad (3.49)$$

and

$$C_3 \equiv v = 0, \quad C_4 \equiv r^2 + k^2 e^{4U} = 0. \quad (3.50)$$

The condition  $C_2 = 0$  is recognized as the Kählerity condition (2.61).

**The Eguchi-Hanson black hole.** The conditions  $C_3 = C_4 = 0$  turn out to be relaxed in the case of the more general Eguchi-Hanson black holes found in [47]:

$$\mathcal{N} = \frac{1}{\sqrt{1 - m^4/\rho^4}}, \quad a = \frac{\rho}{2}, \quad b = \frac{\rho}{2} \sqrt{1 - m^4/\rho^4}, \quad (3.51a)$$

$$\tau_2 = i\tau_1 = f = \left[ \lambda - \frac{\chi^2}{9m^4\rho^2} + \delta \log \left( \frac{\rho^2 - m^2}{\rho^2 + m^2} \right) \right]^{-1}, \quad U^1 = \frac{\chi}{4\sqrt{3}\rho^2}, \quad (3.51b)$$

$$\Psi = \gamma\rho^2 - \frac{\chi\lambda}{4\rho^2} + \frac{\chi^3}{54m^4\rho^4} + \frac{\delta\chi}{4\rho^2m^4} \left[ (\rho^4 - m^4) \log \left( \frac{\rho^2 - m^2}{\rho^2 + m^2} \right) + 2m^2\rho^2 \right] \quad (3.51c)$$

$$\tilde{\zeta}_0 = \text{cte}, \quad \tilde{\zeta}_1 = \text{cte}, \quad \sigma = -\frac{i}{4}\rho^2 - \frac{im^4}{4\rho^2} - \frac{\chi}{6\sqrt{2}\rho^2}\tilde{\zeta}_1 - \frac{i\chi\delta}{2\sqrt{2}m^2}\tilde{\zeta}_0 + \frac{i}{\sqrt{2}}\Psi\tilde{\zeta}_0 \quad (3.51d)$$

where  $\tilde{\zeta}_0$  and  $\tilde{\zeta}_1$  are constants. The conserved charges are given by

$$E_{p^0} = E_{p^1} = 0, \quad E_{q_0} = 2i\tilde{\zeta}_0, \quad E_{q_1} = \frac{2i\tilde{\zeta}_1}{\sqrt{3}}, \quad E_k = -i, \quad Y_0 = 0, \quad (3.52a)$$

$$H = \frac{\chi\delta\tilde{\zeta}_0}{m^2\sqrt{2}}, \quad Y_+ = \frac{1}{12\sqrt{2}m^4}(\chi^2 + 18\delta m^6), \quad F_{q_0} = \frac{i}{4\sqrt{2}}(\chi\lambda - 4\gamma m^4), \quad F_{p^0} = -\frac{i\sqrt{2}}{3}\tilde{\zeta}_1^2, \quad (3.52b)$$

while  $F_{q_1}, F_{p^1}, F$  are too bulky to be displayed. The affine geodesic parameter is related to  $\rho$  by  $\tau = 4\text{arctanh}(\rho^2/m^2)/m^2$ . For regularity in the range  $\rho \geq m$  one must impose  $\delta = 0$  and  $\chi^2 \leq 9\lambda m^6$ . Moreover closed time-like curves at  $r \rightarrow \infty$  can be avoided by taking  $\gamma = 0$ . Note that this is also possible to analytically continue  $m \rightarrow e^{i\pi/4}m$ , in such a way that the solution is regular on the  $\rho > 0$  axis.

For these Eguchi-Hanson black holes, the conditions (3.48) and (3.49) are satisfied, but  $v \neq 0$  and the Noether charge matrix is no longer nilpotent. Instead, its Jordan form in the  $7 \times 7$  matrix representation has one  $3 \times 3$  nilpotent block and two  $2 \times 2$  blocks of the form  $\begin{pmatrix} \pm m^2 & 1 \\ 0 & \pm m^2 \end{pmatrix}$ .

**Constraint analysis.** This motivates us to study the reduction of the one-dimensional dynamical system under the three constraints (3.48) (still restricting to the ungauged case in this section). Using (3.21),(3.22) and (3.23), it is straightforward to check that the three constraints (3.48) are first class, i.e. that they Poisson-commute among each other and with the Hamiltonian constraint (2.53) on the constraint locus. Thus the Hamiltonian system (2.53) admits a consistent reduction to the 12-dimensional symplectic quotient

$$T^*(\mathbb{R}^+ \times \mathcal{M}_3) // \{e = \bar{E} = u = 0\}. \quad (3.53)$$

On this locus, the Hamiltonian (2.53) reduces to

$$H_{\text{WDW}} = N \left( \frac{1}{16} p_r^2 - 1 \right) - \frac{r^2}{N} v \bar{v} = N \left[ \frac{1}{16} \left( p_r^2 - \frac{p_U^2}{r^2} \right) - \frac{e^{4U} k^2}{4r^2} - 1 \right]. \quad (3.54)$$

Moreover, the generators  $E_{p_I}, E_{q_I}, E_k, H, Y_0, Y_{\pm}$  also commute with these constraints, and therefore lead to an action of  $\mathbb{R} \times G_4 \times H_5$  on the phase space (3.53).

Having imposed the constraints (3.48), it may be checked easily that the additional constraint  $\bar{v} = 0$ , commutes with  $e = \bar{E} = u = 0$  as well as with  $H_{\text{WDW}}$ , thus proving the consistency of the constraints (3.33) (or their complex conjugates) relevant for the BMPV and Taub-NUT black holes.

Instead, we want to enforce the constraints  $C_1 = C_2 = 0$  in (3.49), which were found to govern the Gödel and Eguchi-Hanson black holes. Rewriting the constraints (3.49) and the Hamiltonian as

$$C_1 = \frac{e^{-2U}}{N} \left[ \bar{v}(\bar{v} - v) - \frac{N^2}{r^2} \right], \quad C_2 = dr - r\bar{v}, \quad (3.55)$$

$$H_{\text{WDW}} = \frac{1}{N} C_2 (C_2 + 2r\bar{v}) + e^{2U} C_1, \quad (3.56)$$

and using (3.21), it is straightforward to check that the algebra of constraints is first class,

$$\{C_1, u\} = -i \frac{kN}{4r^2} u, \quad \{C_1, \bar{E}\} = -i \frac{kN}{4r^2} \bar{E}, \quad \{C_1, e\} = 0 \quad (3.57a)$$

$$\{C_2, u\} = \frac{N}{4r} u, \quad \{C_2, \bar{E}\} = \frac{N}{4r} \bar{E}, \quad \{C_2, e\} = \frac{N}{2r} e \quad (3.57b)$$

$$\{C_1, C_2\} = -\frac{N}{2r} C_1, \quad \{C_1, H_{\text{WDW}}\} = \{C_2, H_{\text{WDW}}\} = 0. \quad (3.57c)$$

Thus, the Hamiltonian system (2.53) can be further reduced to the 8-dimensional symplectic quotient

$$T^*(\mathbb{R}^+ \times \mathcal{M}_3) // \{e = \bar{E} = u = C_1 = C_2 = 0\}. \quad (3.58)$$

This is the habitat for the Gödel and Eguchi-Hanson solutions (3.46) and (3.51). Note that this phase space is also invariant under  $\mathbb{R} \times G_4 \times H_5$ .

It is worth noting that the phase space (3.58) can be further reduced with respect to the second class constraints (3.50). In this subspace, the Noether charge matrix is nilpotent of degree 7, and for  $k = -i$  the symmetry is enhanced to  $\text{SU}(2) \times \text{SU}(2)$  (as the condition  $C_4 = 0$  is equivalent to  $a/b = \pm ik$ ). This subspace contains the Gödel solution (3.46), as well as the non-spinning BMPV solution (3.30) with  $j = \chi = 0$ .

**The semi-classical radial wave function of Gödel-type black holes.** The five constraints

$$e = \bar{E} = u = C_1 = C_2 = 0 \quad (3.59)$$

are solved in the sector

$$Y_+ = q, \quad E_k = k \quad (3.60)$$

by setting  $p_a = \partial_{\phi^a} S$ , where

$$S_{q,k,f} = -ik e^{2U} + 2i \frac{e^{-2U} r^2}{k} + \sqrt{2} q \tau + k \left[ \sigma + \zeta^0 \tilde{\zeta}_0 + \zeta^1 \tilde{\zeta}_1 + 6\tau \zeta_1^2 - 6\tau^2 \zeta^0 \zeta^1 + 2\tau^3 (\zeta^0)^2 \right] + f \left( \tilde{\zeta}_0 + \tau \tilde{\zeta}_1 + 3\tau^2 \zeta^1 - \tau^3 \zeta^0, \tilde{\zeta}_1 + 6\tau \zeta^1 - 3\tau^2 \zeta^0 \right) \quad (3.61)$$

and  $f$  is an arbitrary function of two variables. Thus, a basis of solutions of the quantum constraints in the semi-classical approximation is given by

$$\Psi_{q,k,p^0,p^1} \sim \exp \left[ i S_{q,k,0} + i p^0 \left( \tilde{\zeta}_0 + \tau \tilde{\zeta}_1 + 3\tau^2 \zeta^1 - \tau^3 \zeta^0 \right) + i p^1 \left( \tilde{\zeta}_1 + 6\tau \zeta^1 - 3\tau^2 \zeta^0 \right) \right] \quad (3.62)$$

Note that the wave function is a holomorphic function of  $\tau$ , as required by the condition  $e = 0$ , and that it flattens out in the near horizon region where  $a, b \rightarrow 0$ .

### 3.4 The Gutowski-Reall black hole

**Constraint analysis.** We now turn to the case of gauged supergravity, and study the consequences of the natural generalization of the constraints (3.48) to the gauged case,

$$\underline{e} = \bar{\underline{E}} = \underline{u} = 0, \quad (3.63)$$

In the presence of gauging, the constraints (3.63) no longer commute with the Hamiltonian, but imply secondary constraints. In particular,

$$\{H_{\text{WDW}}, \underline{u}\} = \frac{3e^{-U}}{4\ell \tau_2^{3/2} r} (\tau_2 - i\tau_1) \left( p_r + \frac{p_U}{r} - \frac{2ike^{2U}}{r} \right) \quad (3.64)$$

$$\{H_{\text{WDW}}, \underline{e}\} = \frac{i\sqrt{3}Ne^{-2U}}{2\ell^2 \tau_2^3} (\tau_2 - i\tau_1) \left( 3N(\tau_2 + 3i\tau_1) - \sqrt{3}e^U \tau_2^{3/2} \ell \underline{E} \right) \quad (3.65)$$

Thus, we impose

$$C_0 \equiv \tau_1 + i\tau_2 = 0 \quad (3.66)$$

which ensures the vanishing of both (3.64) and (3.65). This condition is in fact an integrated version of the condition  $\underline{e} = 0$  where the integration constant has been fixed unambiguously. Enforcing (3.66), the vanishing of

$$\{H_{\text{WDW}}, \bar{\underline{E}}\} = \frac{\sqrt{3}Ne^{-U}}{4\ell r \sqrt{\tau_2}} \left( p_r + \frac{p_U}{r} + \frac{2ike^{2U}}{r} \right) = 0 \quad (3.67)$$

implies the same constraint  $C_2$  as in the ungauged case (3.49),

$$C_2 \equiv r' - rU' - ik \frac{Ne^{2U}}{2r} = 0. \quad (3.68)$$

Finally, requiring that  $H_{\text{WDW}} = 0$  on the constraint locus requires that

$$p_U = -2ik e^{2U} - \frac{4i}{k} e^{-2U} r^2 + \frac{6\sqrt{2} e^{-2U} r^2}{\ell k} \left( p_{\tilde{\zeta}_1} + k\zeta_1 \right) \quad (3.69)$$

Expressing the derivative  $\zeta^{1'}$  in terms of the charge  $p^1 = E_{p^1}$ , and  $p_U$  in terms of  $\bar{u}$ , this may be rewritten as

$$\mathcal{C}_1 \equiv ik\bar{u} - N e^{-2U} - i\sqrt{\frac{3}{2}} \frac{N e^{-2U}}{\ell} p^1 = 0. \quad (3.70)$$

In the limit  $\ell \rightarrow \infty$ , this reduces to the condition  $C_1 = 0$  in (3.49). One may check that the five conditions

$$C_0 = \bar{E} = \underline{u} = \mathcal{C}_1 = C_2 = 0 \quad (3.71)$$

commute, and therefore give a consistent first-class reduction of the phase of  $\mathbb{R} \times \text{U}(1) \times \text{SU}(2)$  symmetric solutions of gauged supergravity. The Hamilton-Jacobi functions satisfying the constraints  $\bar{E} = \underline{u} = \mathcal{C}_1 = C_2 = 0$  on the locus  $C_0 = 0$  are

$$\begin{aligned} S_{k,f} = & -ik e^{2U} + \frac{2i}{k} r^2 e^{-2U} \left( 1 + \frac{3i\sqrt{2}k}{\ell} \zeta_1 \right) \\ & + k \left( \sigma + \tilde{\zeta}_0 \zeta^0 + \tilde{\zeta}_1 \zeta^1 \right) + f \left( \tilde{\zeta}_0, \tilde{\zeta}_1 - \frac{3\sqrt{2}}{k\ell} r^2 e^{-2U} \right) \end{aligned} \quad (3.72)$$

so that a basis of semi-classical wave functions are given by

$$\Psi_{p^0, p^1} \sim \exp \left[ iS_{k,0} + ip^0 \tilde{\zeta}_0 + ip^1 \left( \tilde{\zeta}_1 - \frac{3\sqrt{2}}{k\ell} r^2 e^{-2U} \right) \right] \quad (3.73)$$

Note that the semi-classical wave functions are now independent of  $\tau$ , and flatten out near the horizon  $a, b \rightarrow 0$ . As we now demonstrate shortly, this reduced phase space in fact contains the black hole solution of [6].

**The Gutowski-Reall black hole.** Supersymmetric rotating black holes in minimal  $D = 5$  gauged supergravity were constructed in [6], and extended to general  $D = 5$  gauged supergravity in [7]. In the case of minimal supergravity, the solution is given by

$$\mathcal{N} = 1, \quad a = \alpha\ell \sinh(\rho/\ell), \quad b = 2\alpha^2\ell \sinh(\rho/\ell) \cosh(\rho/\ell), \quad (3.74a)$$

$$f^{-1} = 1 + \frac{4\alpha^2 - 1}{12\alpha^2 \sinh^2(\rho/\ell)}, \quad U_1 = \frac{\sqrt{3}}{\ell} a^2 + \frac{4\alpha^2 - 1}{2\sqrt{3}} \ell \quad (3.74b)$$

$$\Psi = -2\epsilon\alpha^2\ell \sinh^2(\rho/\ell) \left[ 1 + \frac{4\alpha^2 - 1}{4\alpha^2 \sinh^2(\rho/\ell)} + \frac{(4\alpha^2 - 1)^2}{96\alpha^4 \sinh^4(\rho/\ell)} \right]. \quad (3.74c)$$

This may be supplemented with

$$\tilde{\zeta}_0 = 0, \quad \tilde{\zeta}_1 = \frac{3i\sqrt{2}}{\ell} a^2, \quad \sigma = -\frac{i}{2} \alpha^2 \ell^2 \cosh(2\rho/\ell) [1 - 2\alpha^2 + \alpha^2 \cosh(2\rho/\ell)] \quad (3.74d)$$

so as to provide a solution of the motion on  $\mathbb{R}^+ \times G_{2(2)}/\text{SO}(4)$  with  $E_{p^0} = E_{p^1} = 0$  and  $E_k = -i$ . The Noether charges for the symmetry group unbroken by the gauging are given by

$$Y_+ = -\frac{3iR_0^2}{4\sqrt{2}} \left( 1 + \frac{R_0^2}{2\ell^2} \right), \quad F_{q_0} = -\frac{3iR_0^4}{4\sqrt{2}\ell} \left( 1 + \frac{2R_0^2}{3\ell^2} \right) \quad (3.75a)$$

$$H - 2Y_0 = \frac{3\ell^2}{16} \left( 1 + 2\frac{R_0^2}{\ell^2} - 3\frac{R_0^4}{\ell^4} \right), \quad E_{q_0} = E_{q_1} = 0 \quad (3.75b)$$

where the parameter  $\alpha$  was traded for

$$R_0 = \ell \sqrt{\frac{4\alpha^2 - 1}{3}}. \quad (3.76)$$

The charges  $Y_+$  and  $F_{q_0}$  are as usual proportional to the charge and angular momentum,

$$Q_e = \frac{\sqrt{3}\pi R_0^2}{2G} \left(1 + \frac{R_0^2}{2\ell^2}\right) = \frac{2\pi i\sqrt{2}}{G\sqrt{3}} Y_+, \quad J = \frac{3\epsilon\pi R_0^4}{8G\ell} \left(1 + \frac{2R_0^2}{3\ell^2}\right) = \frac{i\pi}{\sqrt{2}G} F_{q_0} \quad (3.77)$$

Note however that the conserved charge  $H - 2Y_0$  differs from the ADM mass

$$M = \frac{3\pi R_0^2}{4G} \left(1 + \frac{3R_0^2}{2\ell^2} + \frac{2R_0^4}{3\ell^4}\right) = \frac{\sqrt{3}}{2} |Q_e| + \frac{2}{\ell} |J|, \quad (3.78)$$

although this could be easily rectified by adding a constant term in  $\sigma$ . Finally, its Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi^2}{2G} \lim_{\rho \rightarrow 0} \frac{a^2 \sqrt{b^2 - f^3 \Psi^2}}{f^{3/2}} = \frac{\pi^2 R_0^3}{2G} \sqrt{1 + \frac{3R_0^2}{4\ell^2}} \quad (3.79)$$

We now check that this solution satisfies the constraints (3.68) and (3.70). In terms of  $\rho$ -derivatives, the latter may be rewritten as

$$U'/N = -\frac{i}{k} e^{-2U} - \frac{ik e^{2U}}{2r^2} + \frac{3\sqrt{2}}{\ell} e^{-2U} \zeta^1 \quad (3.80a)$$

$$(r' - rU')/N = ik \frac{e^{2U}}{2r} \quad (3.80b)$$

Translating to the variables  $a, b, \mathcal{N}, \Psi, U^1$  using (2.60), the conditions  $\bar{E} = 0$  and  $\bar{u} = 0$  become, respectively,

$$a^2 U^{1'} + ikb\mathcal{N}U^1 = \frac{2\sqrt{3}}{\ell} f^{-1} a^2 b \mathcal{N} \quad (3.81a)$$

$$a^2 U^{1'} - ikb\mathcal{N}U^1 = -\frac{f}{\sqrt{3}} (a^2 \Psi' - \mathcal{N}b\Psi) \quad (3.81b)$$

while the conditions (3.70) and (3.68) become

$$\frac{b'}{\mathcal{N}} + ik \frac{b^2}{2a^2} - \frac{1}{ik} = \frac{2\sqrt{3}}{\ell} U^1, \quad ikb\mathcal{N} = 2aa' \quad (3.81c)$$

These four equations agree with eq. (3.1), (3.15), (3.16), (3.17) of [7] when  $k = -i$  and in the gauge  $\mathcal{N} = 1$ , while the constraint (3.66) is identical with the ansatz eq. (3.11) in this same reference. Moreover, solving (3.81c) for  $U^1$  and plugging back in (3.81a), we find eq. (3.12) in [7]

$$f^{-1} = \frac{\ell^2}{12a^2 a'} (4(a')^3 + 7aa'a'' - a' + a^2 a''') . \quad (3.82)$$

Thus, we have recovered the Gutowski-Reall solution via algebraic considerations in the one-dimensional reduced model. We note that the ‘‘flat limit’’  $\ell \rightarrow \infty, \alpha \rightarrow \frac{1}{2}$  keeping  $R_0$  and  $R$  fixed leads back to the the BMPV black hole (3.30) with zero angular momentum. It would be very interesting to find a more general BPS solution where the electric charge  $Q$  and the angular momenta  $J_1, J_2$  can be varied independently. Most likely, this requires going beyond the cohomogeneity one case studied in this paper.

### 3.5 Other solutions of gauged supergravity

We first note that the  $AdS_5$  vacuum solution is obtained by taking  $\alpha = \frac{1}{2}$ , and has  $\bar{\epsilon} = 0$  in addition to the five constraints in (3.71).

Secondly, the near horizon geometry is obtained by taking the limit  $\rho \rightarrow 0$ . Dropping an irrelevant additive constant in  $\sigma$ , we find

$$\begin{aligned} \mathcal{N} = 1, \quad a = \alpha\rho, \quad b = 2\alpha^2\rho, \quad f = \tau_2 = i\tau_1 = \frac{12\alpha^2}{(4\alpha^2 - 1)\ell}\rho^2, \quad U^1 = \frac{(4\alpha^2 - 1)\ell}{2\sqrt{3}}, \\ \Psi = \frac{(4\alpha^2 - 1)^2\ell^3}{48\alpha^2\rho^2}, \quad \tilde{\zeta}_0 = 0, \quad \tilde{\zeta}_1 = \frac{3i\sqrt{2}\alpha^2}{\ell}\rho^2, \quad \sigma = -i\alpha^2\rho^2 \end{aligned} \quad (3.83)$$

It has the same Noether charges as the full solution, and in addition has  $\bar{E} = 0$ . It would be interesting to connect this observation to the enhancement of supersymmetry at the horizon [46].

Thirdly, we note that, upon relaxing the BPS condition, a two-parameter family of  $AdS_2 \times S^3$  geometries with  $E_{pI} = 0$  and  $E_k = -i$  is allowed [50]:

$$\begin{aligned} \mathcal{N} = 1, \quad a = \alpha\rho, \quad b = 2\alpha^2\rho, \quad f = \tau_2 = i\tau_1 = \frac{12\alpha^2\beta}{(4\alpha^2 - 1)\ell}\rho^2, \quad U^1 = \frac{(4\alpha^2 - 1)\ell}{2\sqrt{3}\beta}, \\ \Psi = \frac{(4\alpha^2 - 1)^2\ell^3}{48\alpha^2\beta^2\rho^2}, \quad \tilde{\zeta}_0 = 0, \quad \tilde{\zeta}_1 = \frac{3i\sqrt{2}\alpha^2}{\ell}\rho^2, \quad \sigma = -i\frac{\alpha^2}{\beta}(1 + 4\alpha^2(\beta - 1)\rho^2) \end{aligned} \quad (3.84)$$

This solution still satisfies  $C_0 = \bar{E} = \underline{u} = C_2 = 0$ , but has  $\mathcal{C}_1 \neq 0$  unless  $\beta = 1$ . Its electric charge and angular momentum are now independent parameters,

$$Y_+(\beta) = \frac{Y_+(1)}{\beta^2} - (\beta - 1)\frac{3iR_0^2}{4\sqrt{2}}\left(1 + \frac{3R_0^2}{\ell^2}\right), \quad F_{q_0}(\beta) = \frac{F_{q_0}(1)}{\beta^2} - (\beta - 1)\frac{3iR_0^4}{4\sqrt{2}\ell}\left(1 + \frac{3R_0^2}{\ell^2}\right), \quad (3.85)$$

where  $Y_+(1)$  and  $F_{q_0}(\beta)$  denote the values for the Gutowski-Reall black hole in (3.75). The Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi^2 R_0^3}{2G} \sqrt{1 + \frac{3R_0^2}{4\ell^2} + (\beta - 1)\left(1 + \frac{3R_0^2}{\ell^2}\right)} \quad (3.86)$$

It would be interesting to know if there exists a smooth interpolating solution between this non-BPS extremal geometry and  $AdS_5$  at infinity.

Finally, we note that BPS solutions of  $\mathcal{N} = 1$  supergravity with naked singularities and closed time-like curves were constructed in [51, 52]. In the non-rotating case, their solution is given by

$$\mathcal{N} = \left[1 + \frac{\rho^2}{\ell^2}\left(1 + \frac{q}{r^2}\right)^3\right]^{-1/4}, \quad a = b = \frac{\rho}{2\mathcal{N}}, \quad f = \mathcal{N}^{-2}\left(1 + \frac{q}{r^2}\right)^{-1}, \quad (3.87a)$$

$$\tau_1 = -i\left(1 + \frac{q}{r^2}\right)^{-1}, \quad \Psi = U^1 = \tilde{\zeta}_0 = \tilde{\zeta}_1 = 0, \quad \sigma = -\frac{i}{4}\rho^2 \quad (3.87b)$$

Its conserved charges are

$$Y_+ = -\frac{3i}{4\sqrt{2}}q, \quad E_k = -i, \quad E_{pI} = E_{qI} = F_{q_0} = H - 2Y_0 = 0 \quad (3.88)$$

While the shifted quaternionic viel-bein does not seem to exhibit any particular structure, the unshifted viel-bein satisfies

$$u = \bar{u} = E = \bar{E} = 0, \quad v = \bar{v} - Ne^{-2U}. \quad (3.89)$$

Thus, for this type of solution the contributions  $\mathcal{X}^a B_\mu \gamma^\mu$  and  $g\mathcal{X}^a$  in (2.39) have to cancel. It is straightforward to see that the four conditions  $u = \bar{u} = E = \bar{E} = 0$  commute with the Lagrangian on the constraint locus, however they are not first class since  $v - \bar{v}$  does not vanish. It would be desirable to clarify the nature of this type of BPS solutions.

#### 4. Discussion

In this work we took the first steps in extending the algebraic methods which have been so useful for studying for 4D black holes, to the case of  $D = 5, \mathcal{N} = 1$  supergravity, with and without gauging. In particular, we have constructed the non-linear sigma model arising in the reduction of stationary solutions with a  $U(1)$  isometry to  $D = 3$ , and identified the appropriate gauging. We further studied the reduction to  $D = 1$  appropriate to solutions with  $U(1) \times SU(2)$  isometries, and studied the algebra of conserved charges and supersymmetry constraints. These have been illustrated on a number of known solutions in gauged and ungauged gravity, including the BMPV black hole and its generalizations, the Gödel and Eguchi-Hanson black holes, and the Gutowski-Reall solution.

In the process, we have found evidence for a new supersymmetric completion of the bosonic sigma model, distinct from the one relevant for 4D BPS black holes, and traced its origin to the non-trivial behavior of the Killing spinors along the fibers of the reduction. We have also found that the supersymmetric solutions of the ungauged theory exist in two branches, only one of which seems to subsist at finite  $\ell$ . It would be very interesting to see if more general supersymmetric solutions of gauged supergravity are allowed, where an arbitrary linear combination of the first and second rows of the (shifted) quaternionic viel-bein vanishes, or more generally whether new (SUSY or non-SUSY) solutions may be reached by transformations in  $G_{2(2)}$  which commute with the gauging.

The eventual goal of our construction is to provide a general framework to describe 5D solutions of gauged supergravity, particularly in the BPS sector. While we have concentrated on BPS black hole solutions, with  $\mathbb{R} \times S^3$  conformal boundary, it would be useful to fit the AdS black strings, studied e.g. in [53–57], in our formalism. More ambitiously, it would be very desirable to extend our methods to the co-homogeneity 2 case (relevant for stationary solutions with  $U(1) \times U(1)$  isometries or co-homogeneity 3 case (for solutions with a single isometry), which may allow us to construct new multi-centered black hole or black ring solutions in AdS. We hope to return to some of these problems in future work.

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### A. More general nilpotent solutions of degree 2

The nilpotency condition  $Q^2 = 0$  allows for more general solutions than the Taub-NUT vacuum (3.38) with  $j = \mu = 0$ . In view of the fact that solutions with  $Q^2 = 0$  are associated to the minimal co-adjoint orbit of  $G_3$ , and the conjectural relation to the generalized topological amplitude [58, 59], we briefly present them here, postponing their interpretation to future work

For a given value of the moduli, such solution are uniquely determined by the charges  $p^I, q_I, k$  subject to two conditions, e.g. at the identity of  $G_{2(2)}/SO(4)$

$$3p^0q_1 - \sqrt{3}q_1^2 + \sqrt{3}(p^1)^2 + 3p^1q_0 = 0, \quad k^2 = -\frac{(p^1q_0 - p^0q_1)(3p^0q_0 + p^1q_1)^2}{6\sqrt{3}(p^0p^1 + q_0q_1)}. \quad (\text{A.1})$$

For example, setting  $k = -i, p^0 = \sqrt{2}\mu/\sqrt{1+\mu^2}, q_0 = i\sqrt{2}\mu$ , we find

$$a = \frac{1}{\rho+\gamma} \left(1 - \frac{2(2+\mu^2)}{\sqrt{1+\mu^2}}\rho + 4\rho^2\right)^{1/4}, \quad b = \left(1 - \frac{2(2+\mu^2)}{\sqrt{1+\mu^2}}\rho + 4\rho^2\right)^{-1/4} \quad (\text{A.2a})$$

$$\mathcal{N} = a^2b, \quad f = \sqrt{\frac{1-4\rho^2+\mu^2(1-2\rho(\sqrt{1+\mu^2}+2\rho))}{1+\mu^2-4\rho^2}} \quad (\text{A.2b})$$

$$\Psi = \frac{2\mu\rho}{1-2\rho\sqrt{1+\mu^2}}, \quad \tilde{\zeta}_0 = \frac{\sqrt{2}\mu\rho}{2\rho-\sqrt{1+\mu^2}}, \quad \tau_1 = U^1 = \tilde{\zeta}_1 = 0 \quad (\text{A.2c})$$

$$\sigma = 2i\rho \frac{2\mu^4\rho^2 + (4\rho^2-1)(1+\mu^2 + (2+\mu^2\rho)\sqrt{1+\mu^2})}{(1+\mu^2-4\rho^2)(1-4(1+\mu^2)\rho^2)} \quad (\text{A.2d})$$

where  $\rho$  is equal to the geodesic affine parameter. This solution has conserved charges

$$H = -\frac{2(2+\mu^2)}{\sqrt{1+\mu^2}}, \quad Y_0 = \frac{3\mu^2}{\sqrt{1+\mu^2}}, \quad F_{q^0} = 2i\sqrt{2}\mu, \quad (\text{A.3a})$$

$$E_{p^0} = -\frac{2\sqrt{2}\mu}{\sqrt{1+\mu^2}}, \quad E_{q_0} = 2i\sqrt{2}\mu, \quad E_{p^1} = E_{q_1} = Y_+ = 0. \quad (\text{A.3b})$$

This solution asymptotes to Taub-NUT space at  $\rho \rightarrow -\gamma$ , and has an orbifold singularity at  $\rho \rightarrow \infty$ . Note that it carries no electromagnetic flux in 5 dimensions.

### References

- [1] R. Emparan and H.S. Reall, *Black holes in higher dimensions*, arXiv:0801.3471.
- [2] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [hep-th/9601029].



- [3] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett. B* **391** (1997) 93 [[hep-th/9602065](#)].
- [4] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4D and 5D black holes*, *JHEP* **02** (2006) 024 [[hep-th/0503217](#)].
- [5] D. Gaiotto, A. Strominger and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023 [[hep-th/0504126](#)].
- [6] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS<sub>5</sub> black holes*, *JHEP* **02** (2004) 006 [[hep-th/0401042](#)].
- [7] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS<sub>5</sub> black holes*, *JHEP* **04** (2004) 048 [[hep-th/0401129](#)].
- [8] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys. Rev. D* **72** (2005) 041901 [[hep-th/0505112](#)].
- [9] H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS<sub>5</sub> black holes*, *JHEP* **04** (2006) 036 [[hep-th/0601156](#)].
- [10] H.K. Kunduri, J. Lucietti and H.S. Reall, *Do supersymmetric anti-de Sitter black rings exist?*, *JHEP* **02** (2007) 026 [[hep-th/0611351](#)].
- [11] H.K. Kunduri and J. Lucietti, *Near-horizon geometries of supersymmetric AdS<sub>5</sub> black holes*, *JHEP* **12** (2007) 015 [[arXiv:0708.3695](#)].
- [12] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)].
- [13] M. Berkooz, D. Reichmann and J. Simon, *A Fermi surface model for large supersymmetric AdS<sub>5</sub> black holes*, *JHEP* **01** (2007) 048 [[hep-th/0604023](#)].
- [14] M.J. Duff and J.T. Liu, *Anti-de Sitter black holes in gauged N = 8 supergravity*, *Nucl. Phys. B* **554** (1999) 237 [[hep-th/9901149](#)].
- [15] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Rotating black holes in gauged supergravities: thermodynamics, supersymmetric limits, topological solitons and time machines*, [hep-th/0504080](#).
- [16] M. Bodner and A.C. Cadavid, *Dimensional reduction of type IIB supergravity and exceptional quaternionic manifolds*, *Class. and Quant. Grav.* **7** (1990) 829.
- [17] S. Mizoguchi and N. Ohta, *More on the similarity between D = 5 simple supergravity and M-theory*, *Phys. Lett. B* **441** (1998) 123 [[hep-th/9807111](#)].
- [18] A. Bouchareb et al., *G<sub>2</sub> generating technique for minimal D = 5 supergravity and black rings*, *Phys. Rev. D* **76** (2007) 104032 [[arXiv:0708.2361](#)].
- [19] G. Clement, *The symmetries of five-dimensional minimal supergravity reduced to three dimensions*, [arXiv:0710.1192](#).
- [20] D. Gaiotto, W.W. Li and M. Padi, *Non-supersymmetric attractor flow in symmetric spaces*, *JHEP* **12** (2007) 093 [[arXiv:0710.1638](#)].
- [21] M. Günaydin, A. Neitzke, O. Pavlyk and B. Pioline, *Quasi-conformal actions, quaternionic discrete series and twistors: SU(2,1) and G<sub>2</sub>(2)*, [arXiv:0707.1669](#).

- [22] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and quasiconformal realizations of exceptional Lie groups*, *Commun. Math. Phys.* **221** (2001) 57 [[hep-th/0008063](#)].
- [23] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *BPS black holes, quantum attractor flows and automorphic forms*, *Phys. Rev. D* **73** (2006) 084019 [[hep-th/0512296](#)].
- [24] A. Neitzke, B. Pioline and S. Vandoren, *Twistors and black holes*, *JHEP* **04** (2007) 038 [[hep-th/0701214](#)].
- [25] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *Quantum attractor flows*, *JHEP* **09** (2007) 056 [[arXiv:0707.0267](#)].
- [26] B. Pioline, *Lectures on black holes, topological strings and quantum attractors*, *Class. and Quant. Grav.* **23** (2006) S981 [[hep-th/0607227](#)].
- [27] B. de Wit, I. Herger and H. Samtleben, *Gauged locally supersymmetric  $D = 3$  nonlinear sigma models*, *Nucl. Phys. B* **671** (2003) 175 [[hep-th/0307006](#)].
- [28] B. de Wit, H. Nicolai and H. Samtleben, *Gauged supergravities in three dimensions: a panoramic overview*, [hep-th/0403014](#).
- [29] M. Günaydin, G. Sierra and P.K. Townsend, *The geometry of  $N = 2$  Maxwell-Einstein supergravity and Jordan algebras*, *Nucl. Phys. B* **242** (1984) 244.
- [30] M. Günaydin, G. Sierra and P.K. Townsend, *Gauging the  $D = 5$  Maxwell-Einstein supergravity theories: more on Jordan algebras*, *Nucl. Phys. B* **253** (1985) 573.
- [31] M. Günaydin, G. Sierra and P.K. Townsend, *The  $N = 2$  Maxwell-Einstein supergravity theories: their compact and noncompact gaugings and Jordan algebras*, in G.W. Gibbons, S.W. Hawking and P.K. Townsend, *Supersymmetry and its applications — superstrings, anomalies and supergravity: proceedings*, Cambridge University Press, Cambridge U.K. (1985).
- [32] M. Günaydin, S. McReynolds and M. Zagermann, *Unified  $N = 2$  Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in four dimensions*, *JHEP* **09** (2005) 026 [[hep-th/0507227](#)].
- [33] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. I: vector multiplets*, *JHEP* **03** (2004) 028 [[hep-th/0312001](#)].
- [34] A. Ceresole, S. Ferrara and A. Marrani,  *$4D/5D$  correspondence for the black hole potential and its critical points*, *Class. and Quant. Grav.* **24** (2007) 5651 [[arXiv:0707.0964](#)].
- [35] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Vector multiplets coupled to  $N = 2$  supergravity: superhiggs effect, flat potentials and geometric structure*, *Nucl. Phys. B* **250** (1985) 385.
- [36] L. Andrianopoli et al., *General matter coupled  $N = 2$  supergravity*, *Nucl. Phys. B* **476** (1996) 397 [[hep-th/9603004](#)].
- [37] P. Breitenlohner, D. Maison and G.W. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, *Commun. Math. Phys.* **120** (1988) 295.
- [38] B. de Wit, A.K. Tollsten and H. Nicolai, *Locally supersymmetric  $D = 3$  nonlinear sigma models*, *Nucl. Phys. B* **392** (1993) 3 [[hep-th/9208074](#)].
- [39] D. Maison, *Ehlers-Harrison type transformations for Jordan's extended theory of gravitation*, *Gen. Rel. Grav.* **10** (1979) 717.

- [40] S. Giusto and A. Saxena, *Stationary axisymmetric solutions of five dimensional gravity*, *Class. and Quant. Grav.* **24** (2007) 4269 [[arXiv:0705.4484](#)].
- [41] J. Ford, S. Giusto, A. Peet and A. Saxena, *Reduction without reduction: adding KK-monopoles to five dimensional stationary axisymmetric solutions*, *Class. and Quant. Grav.* **25** (2008) 075014 [[arXiv:0708.3823](#)].
- [42] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.
- [43] H. Nicolai and H. Samtleben, *Chern-Simons vs. Yang-Mills gaugings in three dimensions*, *Nucl. Phys.* **B 668** (2003) 167 [[hep-th/0303213](#)].
- [44] E. Cremmer, S. Ferrara, C. Kounnas and D.V. Nanopoulos, *Naturally vanishing cosmological constant in  $N = 1$  supergravity*, *Phys. Lett.* **B 133** (1983) 61.
- [45] K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kahler manifolds*, *Commun. Math. Phys.* **108** (1987) 117.
- [46] A. Sinha, J. Sonner and N.V. Suryanarayana, *At the horizon of a supersymmetric  $AdS_5$  black hole: isometries and half-BPS giants*, *JHEP* **01** (2007) 087 [[hep-th/0610002](#)].
- [47] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [[hep-th/0209114](#)].
- [48] J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, *Phys. Rev.* **D 68** (2003) 105009 [*Erratum ibid.* **D 70** (2004) 089901] [[hep-th/0304064](#)].
- [49] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Black holes of  $D = 5$  supergravity*, *Class. and Quant. Grav.* **16** (1999) 1 [[hep-th/9810204](#)].
- [50] J.F. Morales and H. Samtleben, *Entropy function and attractors for  $AdS$  black holes*, *JHEP* **10** (2006) 074 [[hep-th/0608044](#)].
- [51] K. Behrndt, A.H. Chamseddine and W.A. Sabra, *BPS black holes in  $N = 2$  five dimensional  $AdS$  supergravity*, *Phys. Lett.* **B 442** (1998) 97 [[hep-th/9807187](#)].
- [52] D. Klemm and W.A. Sabra, *Charged rotating black holes in 5D Einstein-Maxwell-(A)dS gravity*, *Phys. Lett.* **B 503** (2001) 147 [[hep-th/0010200](#)].
- [53] A.H. Chamseddine and W.A. Sabra, *Magnetic strings in five dimensional gauged supergravity theories*, *Phys. Lett.* **B 477** (2000) 329 [[hep-th/9911195](#)].
- [54] D. Klemm and W.A. Sabra, *Supersymmetry of black strings in  $D = 5$  gauged supergravities*, *Phys. Rev.* **D 62** (2000) 024003 [[hep-th/0001131](#)].
- [55] R.B. Mann, E. Radu and C. Stelea, *Black string solutions with negative cosmological constant*, *JHEP* **09** (2006) 073 [[hep-th/0604205](#)].
- [56] Y. Brihaye, E. Radu and C. Stelea, *Black strings with negative cosmological constant: inclusion of electric charge and rotation*, *Class. and Quant. Grav.* **24** (2007) 4839 [[hep-th/0703046](#)].
- [57] A. Bernamonti et al., *Black strings in  $AdS_5$* , *JHEP* **01** (2008) 061 [[arXiv:0708.2402](#)].
- [58] M. Günaydin, A. Neitzke and B. Pioline, *Topological wave functions and heat equations*, *JHEP* **12** (2006) 070 [[hep-th/0607200](#)].
- [59] P. Gao and B. Pioline, *Topological wave functions and the 4D-5D lift*, [arXiv:0803.0562](#).